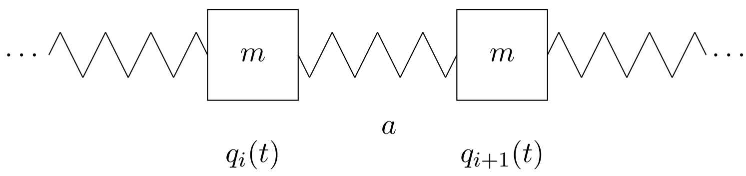
Exercise 1. ***Discrete and continuous models of an elastic rod***

We consider a one-dimensional array of N particles connected by elastic springs with spring force constant κ. Let us assume that all the particles have the same mass m and that, at rest, their relative distance is a. By $q_i(t)$, $i = 1, \ldots N$, we denote their position relative to the equilibrium.



a) Derive the Lagrangian $L(q_i(t), \dot{q}_i(t))$ of this system in the limit of large N.

b) Compute the Euler-Lagrange equations.

c) Take the limit $a \to 0$ of the Lagrangian and the Euler-Lagrange equations that you obtained in a) and b). As in the lecture, denote by $\phi(t, \vec{x})$ the vibration amplitudes, by μ the mass density $\lim_{a\to 0} \frac{m}{a}$, and by Y Young's modulus $\lim_{a\to 0}(\kappa a)$.

d) Check that you obtain the same result if you derive the Euler-Lagrange equations directly from the Lagrangian of an elastic rod.

Exercise 2. ***Equations of motion of electrodynamics***

In electrodynamics the Lagrangian density has the following form:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu$$

where $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$ and j_μ is some external current density.

Using the Euler-Lagrange equations (see lecture 1) and the expression for $\mathcal{L}$ above, derive the equations of motion of the electromagnetic field.

Exercise 3. ***Lorentz transformations***

Consider a classical field $\phi(x)$ and assume its Lagrangian density is invariant under (infinitesimal) Lorentz tranformation $x \to x + \delta x$, with:

$$\delta x^\mu = \omega^\mu_\nu \, x^\nu \tag{1}$$

where ω^μ_ν is an infinitesimal, antisymmetric constant matrix: $\omega^{\mu\nu} = -\omega^{\nu\mu}$. Apply Noether's Theorem to find the conserved current and charge, writing them in terms of the conserved energy-momentum tensor $T^{\mu\nu}$.

Hint. Use the Euler-Lagrange field equations.