Exercise 1. ***Yukawa theory***

Consider a theory with fermions ψ and a real scalar field ϕ coupled through a Yukawa coupling. The Lagrangian reads

$$\mathcal{L} = \bar{\psi}\,(i\,\partial\!\!\!/ - m_0)\,\psi + \frac{1}{2}\,\partial^\mu\phi\,\partial_\mu\phi - \frac{M_0^2}{2}\,\phi^2 - g_0\,\bar{\psi}\psi\phi \tag{1}$$

(a) Find the Feynman rules of this theory and write down the amplitude for the process

$$e^-(p_1)\,e^-(p_2) \to e^-(p_3)\,e^-(p_4)$$

at leading order in perturbation theory.

(b) Compute the differential cross section $d\sigma/d\Omega$ for electron-electron scattering in the Yukawa theory at leading order in perturbation theory.

(c) Rewrite the Lagrangian as $\mathcal{L} = \mathcal{L}_r + \mathcal{L}_{ct}$, where $\mathcal{L}_r$ has the same form as Eq.(1) but is written in terms of renormalized fields, $\psi_R = Z_2^{-1/2}\psi$ and $\phi_R = Z_\phi^{-1/2}\phi$, renormalized masses, m and M and the renormalized coupling g. Write the counterterm Lagrangian $\mathcal{L}_{ct}$ in terms of $\delta_\phi = Z_\phi - 1$, $\delta_M = M_0^2 Z_\phi - M^2$ ...

(d) Calculate the self energy $\Pi(p^2)$ of the scalar field at one loop in renormalized perturbation theory using dimensional regularization.

(e) Use the renormalization conditions

$$\Pi(p^2 = M^2) = 0 \quad \text{and} \quad \frac{d}{dp^2}\Pi(p^2)\big|_{p^2=M^2} = 0$$

to determine the counterterms δ_M and δ_ϕ.

(f) Give an example for a suitable renormalization condition to define the renormalized coupling g.

(g) Is this theory as given in Eq.(1) renormalizable?