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QFT I

Solutions 12.

HS 2013

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Exercise 1. ***Yukawa theory***

Consider a theory with fermions ψ and a real scalar field ϕ coupled through a Yukawa coupling. The Lagrangian reads

$$\mathcal{L} = \bar{\psi}\,(i\,\partial\!\!\!/ - m_0)\,\psi + \frac{1}{2}\,\partial^\mu\phi\,\partial_\mu\phi - \frac{M_0^2}{2}\,\phi^2 - g_0\,\bar{\psi}\psi\phi \tag{1}$$

(a) Find the Feynman rules of this theory and write down the amplitude for the process

$$e^-(p_1)\,e^-(p_2) \to e^-(p_3)\,e^-(p_4)$$

at leading order in perturbation theory.

Solution. The momentum space Feynman rules for this theory are:

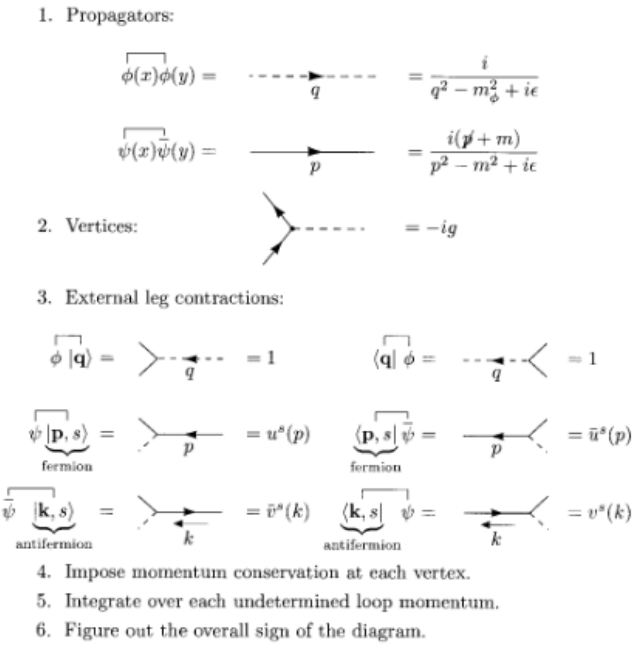

Figure 1: Feynman rules for Yukawa theory [1].

where $m_\phi = M_0$, $m = m_0$ and $g = g_0$ here. At leading order there are two diagrams which contribute to this process; t-channel and u-channel. The amplitudes for these diagrams are:

$$i\mathcal{M}_t = (-ig)^2 \bar{u}_{s_3}(p_3) u_{s_1}(p_1) \left(\frac{i}{t - M_0^2}\right) \bar{u}_{s_4}(p_4) u_{s_2}(p_2)$$
$$i\mathcal{M}_u = -(-ig)^2 \bar{u}_{s_3}(p_3) u_{s_2}(p_2) \left(\frac{i}{u - M_0^2}\right) \bar{u}_{s_4}(p_4) u_{s_1}(p_1)$$

where $t = (p_1 - p_3)^2$, $u = (p_1 - p_4)^2$ and the minus sign in $i\mathcal{M}_u$ comes from the crossing of the final state fermion lines.

(b) Compute the differential cross section $d\sigma/d\Omega$ for electron-electron scattering in the Yukawa theory at leading order in perturbation theory.

Solution. From sheet 9 Ex 1 the differential cross-section (in the centre of mass frame) is:

$$\frac{d\sigma}{d\Omega} = \frac{|\mathcal{M}|^2}{64\pi^2 s}$$

where the Källén function dependence has cancelled because $m_a = m_b = m_c = m_d = m$ and $\mathcal{M}$ is the total amplitude. If one makes the assumption that the initial and final spin states are unknown then one can perform an average of the amplitude over the initial state spins and sum over the final state spins: $\frac{1}{2}\sum_{s_1}\frac{1}{2}\sum_{s_2}\sum_{s_3,s_4}|\mathcal{M}|^2 = \frac{1}{4}\sum_{s_1,s_2,s_3,s_4}|\mathcal{M}|^2 := |\overline{\mathcal{M}}|^2$. For $e^-e^- \to e^-e^-$ one has:

$$|\overline{\mathcal{M}}|^2 = \frac{1}{4}\sum_{spins}\left[|\mathcal{M}_t|^2 + |\mathcal{M}_u|^2 + \mathcal{M}_t\mathcal{M}_u^\dagger + \mathcal{M}_u\mathcal{M}_t^\dagger\right]$$

where the first two terms are (using $\sum_s u_s(p)\bar{u}_s = \not{p} + m$):

$$\begin{aligned}
\frac{1}{4}\sum_{spins}|\mathcal{M}_t|^2 &= \left(\frac{g^4}{4(t-M_0^2)^2}\right)\sum_{spins}\bar{u}_{s_3}(p_3)u_{s_1}(p_1)\bar{u}_{s_4}(p_4)u_{s_2}(p_2)\cdot\bar{u}_{s_2}(p_2)u_{s_4}(p_4)\bar{u}_{s_1}(p_1)u_{s_3}(p_3)\\
&= \left(\frac{g^4}{4(t-M_0^2)^2}\right)\mathrm{Tr}\big[(\not{p}_1+m)(\not{p}_3+m)\big]\mathrm{Tr}\big[(\not{p}_2+m)(\not{p}_4+m)\big]\\
&= \left(\frac{4g^4}{(t-M_0^2)^2}\right)\big[(p_1\cdot p_3)+m^2\big]\big[(p_2\cdot p_4)+m^2\big]\\
&= \left(\frac{4g^4}{(t-M_0^2)^2}\right)\big[(p_1\cdot p_3)(p_2\cdot p_4)+m^2(p_1\cdot p_3)+m^2(p_2\cdot p_4)+m^4\big]\\
\frac{1}{4}\sum_{spins}|\mathcal{M}_u|^2 &= \left(\frac{4g^4}{(u-M_0^2)^2}\right)\big[(p_2\cdot p_3)(p_1\cdot p_4)+m^2(p_2\cdot p_3)+m^2(p_1\cdot p_4)+m^4\big] \qquad \text{(similarly)}
\end{aligned}$$

and the interference terms have the form:

$$\begin{aligned}
\frac{1}{4}\sum_{spins}\big[\mathcal{M}_t\mathcal{M}_u^\dagger + \mathcal{M}_u\mathcal{M}_t^\dagger\big] &= -\left(\frac{g^4}{4(t-M_0^2)(u-M_0^2)}\right)\mathrm{Tr}\big[(\not{p}_1+m)(\not{p}_4+m)(\not{p}_2+m)(\not{p}_3+m)\big]\\
&\quad -\left(\frac{g^4}{4(u-M_0^2)(t-M_0^2)}\right)\mathrm{Tr}\big[(\not{p}_1+m)(\not{p}_3+m)(\not{p}_2+m)(\not{p}_4+m)\big]\\
&= -\left(\frac{2g^4}{(t-M_0^2)(u-M_0^2)}\right)\big[(p_1\cdot p_4)(p_2\cdot p_3)-(p_1\cdot p_2)(p_3\cdot p_4)+(p_1\cdot p_3)(p_2\cdot p_4)\\
&\quad + m^2(p_1\cdot p_4)+m^2(p_1\cdot p_2)+m^2(p_1\cdot p_3)+m^2(p_2\cdot p_3)\\
&\quad + m^2(p_2\cdot p_4)+m^2(p_3\cdot p_4)+m^4\big]
\end{aligned}$$

By making use of the following Mandelstam identities:

$$\begin{aligned}
&(p_1\cdot p_2) = (p_3\cdot p_4) = \frac{s}{2}-m^2, \qquad (p_1\cdot p_3) = (p_2\cdot p_4) = m^2-\frac{t}{2}\\
&(p_1\cdot p_4) = (p_2\cdot p_3) = m^2-\frac{u}{2}, \qquad s+t+u = 4m^2
\end{aligned}$$

the spin averaged/summed amplitude can be written:

$$\begin{aligned}
|\overline{\mathcal{M}}|^2 &= \frac{4g^4}{(t-M_0^2)^2}\left[\left(m^2-\frac{t}{2}\right)^2+2m^2\left(m^2-\frac{t}{2}\right)+m^4\right]+\frac{4g^4}{(u-M_0^2)^2}\left[\left(m^2-\frac{u}{2}\right)^2+2m^2\left(m^2-\frac{u}{2}\right)+m^4\right]\\
&\quad -\frac{2g^4}{(t-M_0^2)(u-M_0^2)}\left[\left(m^2-\frac{u}{2}\right)^2-\left(\frac{s}{2}-m^2\right)^2+\left(m^2-\frac{t}{2}\right)^2+2m^2\left(m^2-\frac{u}{2}\right)\right.\\
&\quad\quad \left.+2m^2\left(m^2-\frac{t}{2}\right)+2m^2\left(\frac{s}{2}-m^2\right)+m^4\right]\\
&= \frac{g^4(t-4m^2)^2}{(t-M_0^2)^2}+\frac{g^4(u-4m^2)^2}{(u-M_0^2)^2}+\frac{g^4(ut-4sm^2)}{(t-M_0^2)(u-M_0^2)}
\end{aligned}$$

hence one has:

$$\frac{d\sigma}{d\Omega} = \frac{g^4}{64\pi^2 s}\left[\frac{(t-4m^2)^2}{(t-M_0^2)^2}+\frac{(u-4m^2)^2}{(u-M_0^2)^2}+\frac{(ut-4sm^2)}{(t-M_0^2)(u-M_0^2)}\right]$$

(c) Rewrite the Lagrangian as $\mathcal{L} = \mathcal{L}_r + \mathcal{L}_{ct}$, where $\mathcal{L}_r$ has the same form as Eq.(1) but is written in terms of renormalized fields, $\psi_R = Z_2^{-1/2}\psi$ and $\phi_R = Z_\phi^{-1/2}\phi$, renormalized masses, m and M and the renormalized coupling g. Write the counterterm Lagrangian $\mathcal{L}_{ct}$ in terms of $\delta_\phi = Z_\phi - 1$, $\delta_M = M_0^2 Z_\phi - M^2$ ...

Solution. Plugging in the expressions for the fields in terms of renomalized ones:

$$\psi = Z_2^{1/2}\psi_R, \quad \phi = Z_\phi^{1/2}\phi_R$$

the Lagrangian becomes:

$$\mathcal{L} = Z_2\,\bar{\psi}_R\,(i\,\not\partial - m_0)\,\psi_R + \frac{1}{2}\,Z_\phi\,\partial^\mu\phi_R\,\partial_\mu\phi_R - \frac{M_0^2}{2}\,Z_\phi\,\phi_R^2 - g_0\,Z_2\,Z_\phi^{1/2}\,\bar{\psi}_R\psi_R\phi_R$$

We still have the bare masses and the bare coupling in this Lagrangian. Now with the definitions $\delta_\phi = Z_\phi - 1$, $\delta_M = M_0^2 Z_\phi - M^2$, $\delta_2 = Z_2 - 1$, $\delta_m = Z_2 m_0 - m$, $g\,Z_1 = g_0\,Z_2\,Z_\phi^{1/2}$ and $\delta_1 = Z_1 - 1$ we can rewrite the Lagrangian as:

$$\begin{aligned}\mathcal{L} =& \bar{\psi}_R\,(i\,\not\partial - m)\,\psi_R + \frac{1}{2}\,\partial^\mu\phi_R\,\partial_\mu\phi_R - \frac{1}{2}\,M^2\,\phi_R^2 - g\,\bar{\psi}_R\psi_R\phi_R\\ &+ \frac{1}{2}\,\delta_\phi\,\partial^\mu\phi_R\,\partial_\mu\phi_R - \frac{1}{2}\,\delta_M\,\phi_R^2 + \bar{\psi}_R\,(i\delta_2\,\not\partial - \delta_m\,)\,\psi_R - g\delta_1\,\bar{\psi}_R\psi_R\phi_R\\ =& \mathcal{L}_r + \mathcal{L}_{ct}\end{aligned}$$

(d) Calculate the self energy $\Pi(p^2)$ of the scalar field at one loop in renormalized perturbation theory using dimensional regularization.

Solution. The scalar field propagator receives corrections at order g^2 from a fermion loop diagram and two propagator counterterms:

$$\begin{aligned}i\Pi(p^2) &= i\,\Pi_2(p^2) + i\left(p^2\delta_\phi - \delta_M\right)\\ &= -(-ig)^2\int\frac{d^Dk}{(2\pi)^D}\mathrm{Tr}\Big[\frac{i(\not k + \not p + m)i(\not k + m)}{[(k+p)^2 - m^2][k^2 - m^2]}\Big] + i\left(p^2\delta_\phi - \delta_M\right)\\ &= -4g^2\int\frac{d^Dk}{(2\pi)^D}\,\frac{k\cdot(p+k) + m^2}{[(k+p)^2 - m^2][k^2 - m^2]} + i\left(p^2\delta_\phi - \delta_M\right)\end{aligned}\tag{S.1}$$

Now we need to bring this integral into the form:

$$\frac{d^Dk}{(2\pi)^D}\,\frac{k^2 + \Delta}{(k^2 - \Delta + i\epsilon)^2}$$

So that we can use the formulas (Series 11, Ex1):

$$\begin{aligned}\frac{d^Dk}{(2\pi)^D}\,\frac{\Delta}{(k^2 - \Delta + i\epsilon)^2} &= \frac{i}{(4\pi)^{D/2}}\Gamma(2 - D/2)\Delta^{D/2-1}\\ \frac{d^Dk}{(2\pi)^D}\,\frac{k^2}{(k^2 - \Delta + i\epsilon)^2} &= \frac{iD/2}{(4\pi)^{D/2}}\Gamma(1 - D/2)\Delta^{D/2-1}\end{aligned}$$

To do this we use the Feynman parametrization to combine the denominators into a single denominator. Then rotate to Euclidean space, and also we shift the loop momentum as:

$$k \to k + xp$$

Then the Equation S.1 becomes:

$$\begin{aligned}i\Pi_2(p^2) &= -4g^2\int_0^1\int dx\frac{d^Dk}{(2\pi)^D}\,\frac{k^2 - x(1-x)p^2 + m^2}{(k^2 + x(1-x)p^2 - m^2)^2}\\ &= -\,4g^2\int_0^1\int dx\frac{-i}{(4\pi)^{D/2}}\left(\frac{D/2\Gamma(1 - D/2)}{\Delta^{1-D/2}} - \frac{\Delta\Gamma(2 - D/2)}{\Delta^{2-D/2}}\right)\\ &= \frac{4ig^2(D-1)}{(4\pi)^{D/2}}\int_0^1\int dx\frac{\Gamma(1 - D/2)}{\Delta^{1-D/2}}\end{aligned}$$

where

$$\Delta = m^2 - x(1-x)p^2$$

(e) Use the renormalization conditions

$$\Pi(p^2 = M^2) = 0 \quad \text{and} \quad \frac{d}{dp^2}\Pi(p^2)\big|_{p^2=M^2} = 0$$

to determine the counterterms δ_M and δ_ϕ.

Solution. In order to satisfy the renormalization conditions both of the counterterms must be nonzero. To determine δ_M we subtract the value of the loop diagram at $p^2 = m^2$ so that:

$$\delta_M = \frac{4g^2(D-1)}{(4\pi)^{D/2}} \int_0^1 dx \frac{\Gamma(1-D/2)}{[m^2 - x(1-x)M^2]^{1-D/2}} + M^2\delta_\phi$$

To determine δ_ϕ we cancel also the first derivative with respect to p^2 of the loop integral. We obtain:

$$\begin{aligned} \delta_\phi &= -\frac{4g^2(D-1)}{(4\pi)^{D/2}} \int_0^1 dx \frac{x(1-x)\Gamma(2-D/2)}{[m^2 - x(1-x)M^2]^{2-D/2}} \\ &\underbrace{\rightarrow}_{D\to 4} -\frac{3g^2}{4\pi^2} \int_0^1 dx x(1-x) \left(\frac{1}{\epsilon} - \gamma - \frac{2}{3} + log(4\pi) - log\left[m^2 - x(1-x)M^2\right]\right) \end{aligned}$$

To write it in terms of Z_M note that these relations actually hold, namely,

$$\begin{aligned} \Pi(M^2) = 0 &= \Pi_2(M^2) - M^2\,\delta_\phi + \delta_M \\ &= \Pi_2(M^2) - (1 - Z_M)\,M^2\,Z_\phi \end{aligned} \tag{S.2}$$

and hence

$$\Pi_2(M^2) = (1 - Z_M)\,M^2 + O(g^4) \tag{S.3}$$

(f) Give an example for a suitable renormalization condition to define the renormalized coupling g.

Solution. A renormalization condition for the vertex could be:

$$-i\Gamma(p_f - p_i = q)\big|_{q^2=M^2} = g \tag{S.4}$$

(g) Is this theory as given in Eq.(1) renormalizable?

Solution. The theory as given in eq (1) is not renormalizable, since box diagrams where fermions run in the loop and external legs are scalar fields also appear, which turn out to be divergent. Therefore one must consider a counterterm for these diagrams and hence the 4-point self interaction term $\frac{\lambda}{4!}\phi^4$ must be added to the Lagrangian.

References

[1] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory*, Addison-Wesley Publishing Company (1995), pp 118.