Exercise 1. *Bilinear covariants*

Transformation of the Dirac equation $(i\partial_\mu \gamma^\mu - m)\psi(x) = 0$ into a different frame related by a Lorentz transformation $x' = \Lambda\, x$ results in $(i\partial'_\mu \gamma^\mu - m)\psi'(x') = 0$ with $\psi'(x') = S(\Lambda)\psi(x)$.

(a) Show that this requirement of form invariance entails that the transformation matrix S has to satisfy $S^{-1}(\Lambda)\gamma^\mu S(\Lambda) = \Lambda^\mu{}_\nu \gamma^\nu$.

(b) Using the explicit form $S(\Lambda) = e^{-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}}$, find the transformation of the Dirac adjoint $\bar{\psi}$.

(c) Determine the transformation of

$$\bar{\psi}\psi\,; \qquad \bar{\psi}\gamma^5\psi\,; \qquad \bar{\psi}\gamma^\mu\psi\,; \qquad \bar{\psi}\gamma^5\gamma^\mu\psi\,; \qquad \bar{\psi}\sigma^{\mu\nu}\psi\,;$$

under (proper orthochronous) Lorentz transformations and parity.

Exercise 2. *Solutions to the Dirac equation*

(a) Starting from the Dirac equation show that the Hamiltonian can be written as

$$H = \begin{pmatrix} m & -i\vec{\sigma}\cdot\vec{\nabla} \\ -i\vec{\sigma}\cdot\vec{\nabla} & -m \end{pmatrix}$$

where the Dirac representation of the γ-matrices has been used. Show that H has two positive and two negative (energy) eigenvalues.

(b) Show that the two ($s \in \{1,2\}$) positive and negative energy eigenfunctions of H can be written as

$$\psi_{+,s}(x) = \begin{pmatrix} \sqrt{E+m}\,\chi_s \\ \frac{\vec{\sigma}\cdot\vec{p}}{\sqrt{E+m}}\,\chi_s \end{pmatrix} e^{-ipx} \quad \text{and} \quad \psi_{-,s}(x) = \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{\sqrt{E+m}}\,\chi_s \\ \sqrt{E+m}\,\chi_s \end{pmatrix} e^{+ipx}$$

where

$$\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\,,\ \chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(c) Consider the special case $\vec{p} = (0,0,p)$. Show that in this case $\psi_{\pm,s}(x)$ are also eigenstates of the spin operator S_z and find the eigenvalues.

Exercise 3. *Majorana field*

(a) Show that for a scalar field $\phi(x)$ and a vector field $A_\mu(x)$ the reality condition $\phi(x) = \phi^*(x)$ and $A_\mu(x) = A^*_\mu(x)$ are Lorentz invariant. Why is this not the case for a spinor field $\psi(x)$?

(b) Show that the Majorana condition for a Dirac spinor $\psi(x) = \psi^c(x)$ with $\psi^c(x) \equiv i\gamma^2\,\psi^*(x)$ is Lorentz invariant and that $(\psi^c)^c = \psi$.