Exercise 1. *Gupta-Bleuler quantization*

Choosing $p^\mu = (E, 0, 0, E)$, we define $|\psi_{sl}\rangle \equiv (a^\dagger_{\vec{p},0} - a^\dagger_{\vec{p},3})|0\rangle$. A general transverse 1-particle state $|\psi_T\rangle$ is an arbitrary linear combination $(c_1 a^\dagger_{\vec{p},1} + c_2 a^\dagger_{\vec{p},2})|0\rangle$.

A state $|\psi'_T\rangle$ is said to be equivalent to $|\psi_T\rangle$ if $|\psi'_T\rangle = |\psi_T\rangle + \kappa|\psi_{sl}\rangle$ with a constant κ and we write $|\psi'_T\rangle \sim |\psi_T\rangle$.

(a) Show $\langle\psi'_T|\psi'_T\rangle = \langle\psi_T|\psi_T\rangle \geq 0$

(b) Show $\langle\psi'_T|P^\mu|\psi'_T\rangle = \langle\psi_T|P^\mu|\psi_T\rangle$ and $\langle\psi'_T|H|\psi'_T\rangle \geq 0$

(c) Show that for any physical state $|\psi_{\text{phys}}\rangle$ we have $\langle\psi_{\text{phys}}|\psi'_T\rangle = \langle\psi_{\text{phys}}|\psi_T\rangle$

Exercise 2. *Photon propagator*

(a) In the lecture the propagator in the Coulomb gauge has been given as

$$D^{\text{tr}}_{\mu\nu}(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip(x-y)}}{p^2 + i0^+} \sum_{\lambda=1}^{2} \epsilon_\mu(p,\lambda)\epsilon^*_\nu(p,\lambda)$$

Show that this can be written as

$$D^{\text{tr}}_{\mu\nu}(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip(x-y)}}{p^2 + i0^+} \left(-g_{\mu\nu} - \frac{p^2 n_\mu n_\nu - (pn)(p_\mu n_\nu + p_\nu n_\mu) + p_\mu p_\nu}{(pn)^2 - p^2}\right)$$

where $n^\mu \equiv (1, 0, 0, 0)$.

(b) Show that the photon propagator in the covariant gauge with arbitrary gauge parameter ξ is given by

$$D_{\mu\nu}(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip(x-y)}}{p^2 + i0^+} \left(-g_{\mu\nu} + (1-\xi)\frac{p_\mu p_\nu}{p^2}\right)$$

Hint: recall that the propagator is the Green function of the equations of motion.

Exercise 3. *Positronium*

Positronium is a bound state of an electron and a positron. It can have total spin $S = 0$ (para-positronium) or $S = 1$ (ortho-positronium).

(a) Show that the parity of a positronium with orbital angular momentum L is $P = (-1)^{L+1}$.

(b) Show that a positronium with angular momentum L and spin S is an eigenstate of the charge conjugation operator with eigenvalue $C = (-1)^{L+S}$.

(c) Positronium can decay into a final state with only photons. Find the minimal number of photons in the final state for the decay of the ground state ($L = 0$) of para-positronium and ortho-positronium respectively.