Quantum Information Theory
Series 2

Exercise 2.1 Information and Description Length

Suppose we have a collection of 12 balls, all identical except that one is either lighter or heavier than the rest. At our disposal is a two-pan balance onto which we can place any number of balls in the left pan and the same number in the right pan. The balance registers which of the pans is heavier, or that they are equal. What is the fewest number of weighings needed to determine the odd ball and whether it is lighter or heavier? It might be useful to consider the following questions:

a) How much information is gained upon learning (i) the state of a flipped coin; (ii) the states of two flipped coins; (iii) the outcome of the roll of a four-sided die?

b) How much information is gained when the odd ball and its weight are identified?

c) How much information is gained on the first step if six balls are weighed against the other six? How much is gained by first weighing four against another four, leaving the rest aside?

d) For your prospective weighing strategy, draw a tree showing the possible outcomes of the chosen weighing and what weighing is to be performed next. At each node, how much information has been gained and how much remains to be gained?

Exercise 2.2 Mutual Information

After losing a bet with your Scottish grandfather about whether listening to the radio forecast would help you predict the weather, you have been studying information theory compulsively to try to come up with a clever argument that would make him stop mocking you. You are convinced that even though you did not guess correctly more often than he, you somehow have more *information* about the weather than he does.

a) The mutual information between two random variables is given by

$$I(X:Y)_P = H(X)_P - H(X|Y)_P,$$

where $H(X)$ is the Shannon entropy of X,

$$H(X)_P = \langle -\log P_X(x)\rangle_x = -\sum_x P_X(x)\log P_X(x)$$

and $H(X|Y)$ is the conditional Shannon entropy of X given Y,

$$H(X|Y)_P = \langle -\log P_{X|Y=y}(x)\rangle_{x,y} = -\sum_{x,y} P_{XY}(x,y)\ \log P_{X|Y=y}(x) = H(XY)_P - H(Y)_P.$$

Compute the mutual information between your guess and the actual weather, and do the same for your grandfather. Remember that your grandfather knows it rains on 80% of the days. You also listen to the forecst, knowing it is right 80% of the time and always correct when it predicts rain.

b) You devise the following betting game to prove that your extra information is useful. You and your grandfather start with £1. Every night each of you can bet part of your money on the next day's weather. If your guess was right you double the amount you bet (e.g., in the first night your grandfather bets £0.2 on rain; if it rains he ends up with £1.2, otherwise with £0.8). Any winnings can be used in future rounds.

What would your strategy be? And your grandfather's? After N days, what is the expected gain for each of you? And what is the probability that he finishes with more money than you?

Exercise 2.3 Channel capacity

a) The asymptotic channel capacity is given by

$$C = \max_{P_X} I(X:Y).$$

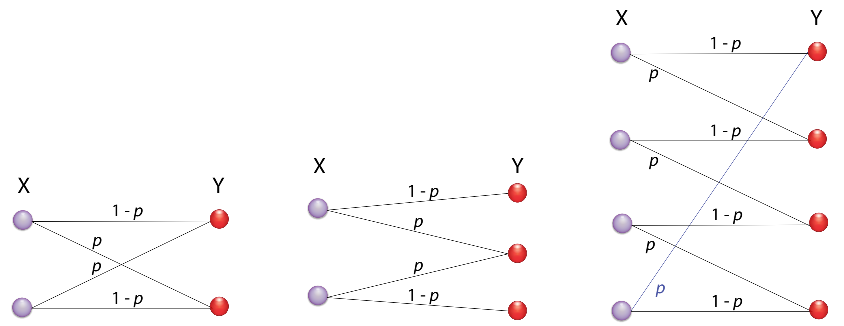
Calculate the asymptotic capacities of the first two channels depicted in Figure 1.

b) We can exploit the symmetries of some channels to simplify the calculation of the capacity.

Consider N possible probability distributions as input to a general channel, $\left\{P_X^i\right\}_i$, with the property that $I(X:Y)_{P^i} = I(X:Y)_{P^j}, \forall i,j$. Suppose you choose which distribution to use for the input by checking a random variable, B, with possible values $b = \{1,\ldots,N\}$. Show that $I(X:Y|B) \leq I(X:Y)$.

How can you use that to find the probability distribution P_X that maximises the mutual information for symmetric channels? **Hint:** consider $\left\{P_X^i\right\}_i$ permutations of P_X^1.

c) Using the result from *b)*, compute the capacity of the last channel. How would you proceed to reliably transmit one bit of information?



(a) Binary Symmetric Channel (b) Symmetric Erasure Channel (c) Yet Another Channel

Figure 1: Three different discrete memoryless channels