Quantum Information Theory
Series 5

HS 13
Dr. J.M. Renes

Exercise 5.1 Bloch sphere

We keep going over some basics of quantum mechanics. In this exercise we will see how we may represent qubit states as points in a three-dimensional ball.
A qubit is a two level system, whose Hilbert space is equivalent to $\mathbb{C}^2$. The Pauli matrices together with the identity form a basis for 2×2 Hermitian matrices,

$$\mathcal{B} = \left\{ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}, \tag{1}$$

where the matrices were represented in basis $\{|0\rangle, |1\rangle\}$. Pauli matrices respect the commutation relations

$$[\sigma_i, \sigma_j] := \sigma_i\sigma_j - \sigma_j\sigma_i = 2\varepsilon_{ijk}\sigma_k, \tag{2}$$
$$\{\sigma_i, \sigma_j\} := \sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}\mathbb{1}. \tag{3}$$

We will see that density operators can always be expressed as

$$\rho = \frac{1}{2}(\mathbb{1} + \vec{r} \cdot \vec{\sigma}) \tag{4}$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ and $\vec{r} = (r_x, r_y, r_z), |\vec{r}| \leq 1$ is the so-called Bloch vector, that gives us the position of a point in a unit ball. The surface of that ball is usually known as the Bloch sphere.

1. Using (4) :
   1) Find and draw in the ball the Bloch vectors of a fully mixed state and the pure states that form three bases, $\{|0\rangle, |1\rangle\}$, $\{|+\rangle, |-\rangle\}$ and $\{|\circlearrowleft\rangle, |\circlearrowright\rangle\}$. Use $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$ and $|\circlearrowleft / \circlearrowright\rangle = (|0\rangle \pm i|1\rangle)/\sqrt{2}$.
   2) Find and diagonalise the states represented by Bloch vectors $\vec{r}_1 = (\frac{1}{2}, 0, 0)$ and $\vec{r}_2 = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$.
2. Show that the operator ρ defined in (4) is a valid density operator for any vector $\vec{r}$ with $|\vec{r}| \leq 1$ by proving it fulfils the following properties:
   1) Hermiticity: $\rho = \rho^\dagger$.
   2) Positivity: $\rho \geq 0$.
   3) Normalisation: $\mathrm{Tr}(\rho) = 1$.
3. Now do the converse: show that any two-level density operator may be written as (4).
4. Check that the surface of the ball is formed by all the pure states.
5. Discuss the analog of the Bloch sphere in higher dimensions. What can be said? For instance, where are the pure states?

Exercise 5.2 The Hadamard Gate

An important qubit transformation in quantum information theory is the Hadamard gate. In the basis of $\sigma_{\hat{z}}$, it takes the form

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{5}$$

That is to say, if $|0\rangle$ and $|1\rangle$ are the $\sigma_{\hat{z}}$ eigenstates, corresponding to eigenvalues $+1$ and -1, respectively, then

$$H = \frac{1}{\sqrt{2}} \left(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1| \right) \tag{6}$$

1. Show that H is unitary.
2. What are the eigenvalues and eigenvectors of H?
3. What form does H take in the basis of $\sigma_{\hat{x}}$? $\sigma_{\hat{y}}$?
4. Give a geometric interpretation of the action of H in terms of the Bloch sphere.