Quantum Information Theory
Series 8

HS 13
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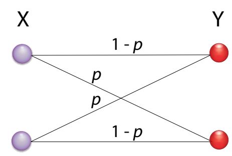
Exercise 8.1 Classical channels as trace-preserving completely positive maps.

In this exercise we will see how to represent classsical channels as trace-preserving completely positive maps (TPCPMs).[1]

a) Take the binary symmetric channel $\mathbf{p}$,



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Recall that we can represent the probability distributions on both ends of the channel as quantum states in a given basis: for instance, if $P_X(0) = q, P_X(1) = 1 - q$, we may express this as the 1-qubit mixed state $\rho_X = q\ |0\rangle\langle 0| + (1-q)\ |1\rangle\langle 1|$.

What is the quantum state ρ_Y that represents the final probability distribution P_Y in the computational basis?

b) Now we want to represent the channel as a map

$$\mathcal{E}_\mathbf{p} : \mathcal{S}(\mathcal{H}_X) \mapsto \mathcal{S}(\mathcal{H}_Y)$$
$$\rho_X \to \rho_Y.$$

An operator-sum representation (also called the Kraus-operator representation) of a CPTP map $\mathcal{E} : \mathcal{S}(\mathcal{H}_X) \to \mathcal{S}(\mathcal{H}_Y)$ is a decomposition $\{E_k\}_k$ of operators $E_k \in \text{Hom}(\mathcal{H}_X, \mathcal{H}_Y)$, $\sum_k E_k {E_k}^\dagger = \mathbb{1}$, such that

$$\mathcal{E}(\rho_X) = \sum_k E_k \rho_X {E_k}^\dagger.$$

Find an operator-sum representation of $\mathcal{E}_\mathbf{p}$.

Hint: think of each operator $E_k = E_{xy}$ as the representation of the branch that maps input x to output y.

c) Now we have a representation of the classical channel in terms of the evolution of a quantum state. What happens if the initial state ρ_X is not diagonal in the computational basis?

d) Now consider an arbitrary classical channel $\mathbf{p}$ from an n-bit space X to an m-bit space Y, defined by the conditional probabilities $\left\{P_{Y|X=x}(y)\right\}_{xy}$.

Express $\mathbf{p}$ as a map $\mathcal{E}_\mathbf{p} : \mathcal{S}(\mathcal{H}_X) \to \mathcal{S}(\mathcal{H}_Y)$ in the operator-sum representation.

[1] Sometimes there are also called completely positive trace-preserving maps (CPTPMs).

Exercise 8.2 Different Quantum Channels

Consider two single-qubit Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ and a CPTP map

$$\begin{aligned}\mathcal{E}_p : \mathcal{S}(\mathcal{H}_A) &\mapsto \mathcal{S}(\mathcal{H}_B)\\ \rho &\to p\frac{\mathbb{1}}{2} + (1-p)\rho,\end{aligned}$$

which is called *depolarizing channel*.

a) Find a Kraus representation for $\mathcal{E}_p$.

Hint: Remember that $\rho \in \mathcal{S}(\mathcal{H}_A)$ can be written in the Bloch sphere representation:

$$\rho = \frac{1}{2}(\mathbb{1} + \vec{r}\cdot\vec{\sigma}), \quad \vec{r}\in\mathbb{R}^3, \quad |\vec{r}| \le 1, \quad \vec{r}\cdot\vec{\sigma} = r_x\sigma_x + r_y\sigma_y + r_z\sigma_z, \tag{1}$$

where σ_x, σ_y and σ_z are Pauli matrices. It may be useful to show that

$$\mathbb{1} = \frac{1}{2}(\rho + \sigma_x\rho\sigma_x + \sigma_y\rho\sigma_y + \sigma_z\rho\sigma_z).$$

b) What happens to the Bloch radius $\vec{r}$ of the initial state when we apply $\mathcal{E}_p$? How can this be interpreted?

c) Find Kraus representations for the following qubit channels

(i) The dephasing channel: $\rho \to \rho' = \mathcal{E}(\rho) = (1-p)\rho + p\,\mathrm{diag}(\rho_{00}, \rho_{11})$ (the off-diagonal elements are annihiliated with probability p).

(ii) The amplitude damping (damplitude) channel, defined by the action $|00\rangle \to |00\rangle$, $|10\rangle \to \sqrt{1-p}|10\rangle + \sqrt{p}|01\rangle$.

Exercise 8.3 Classical capacity of the depolarizing channel

Consider the depolarizing channel we have treated in the exercise before that is described by the CPTP map

$$\begin{aligned}\mathcal{E}_p : \mathcal{S}(\mathcal{H}_A) &\mapsto \mathcal{S}(\mathcal{H}_B)\\ \rho &\to p\frac{\mathbb{1}}{2} + (1-p)\rho.\end{aligned}$$

a) Now we will see what happens when we use this quantum channel to send classical information. We start with an arbitrary input probability distribution $P_X(0) = q, P_X(1) = 1-q$. We encode this distribution in a state $\rho_X = q\ |0\rangle\langle 0| + (1-q)|1\rangle\langle 1|$. Now we send ρ_X over the quantum channel, i.e., we let it evolve under $\mathcal{E}_{\mathbf{p}}$. Finally, we measure the output state, $\rho_Y = \mathcal{E}_{\mathbf{p}}(\rho_X)$ in the computational basis.

Compute the conditional probabilities $\left\{P_{Y|X=x}(y)\right\}_{xy}$.

b) Maximize the mutual information over q to find the classical channel capacity of the depolarizing channel.

c) What happens to the channel capacity if we measure the final state in a different basis?