Quantum Information Theory
Series 11

Exercise 11.1 Information measures bonanza

Take a system A in state ρ. Non-conditional quantum min- and max-entropies are given by

$$H_{\min}(A)_\rho = -\log \max_{\lambda \in \mathrm{spec}(\rho)} \lambda, \qquad H_{\max}(A)_\rho = \log \mathrm{rank}(\rho).$$

For instance, if ρ_A has eigenvalues $\mathrm{spec}(\rho_A) = \{0.6, 0.2, 0.2, 0\}$, we have

$$H_{\min}(A)_\rho = -\log 0.6 \text{ and } H_{\max}(A)_\rho = \log 3$$

The mutual information measures correlations between two systems. For ρ_{AB}, we have

$$\begin{aligned} I(A:B)_\rho &= H(A)_\rho + H(B)_\rho - H(AB)_\rho \\ &= H(A)_\rho - H(A|B)_\rho. \end{aligned}$$

Show that if $\mathrm{spec}(\rho) \prec \mathrm{spec}(\tau)$, then the entropy of ρ is larger than or equal to the entropy of τ, for the von Neumann, min- and max-entropies. $\mathrm{spec}(\rho) \prec \mathrm{spec}(\tau)$ means that $\mathrm{spec}(\tau)$ majorizes $\mathrm{spec}(\rho)$. See exercise 7.3 for more details.

Exercise 11.2 Davies' Theorem

Consider an arbitrary CQ state $\sigma^{XB} = \sum_x p_x |x\rangle\langle x|^X \otimes \rho_x^B$ and imagine making a measurement $\mathcal{M}$ having elements E_y on B. By the Holevo bound, $I(X{:}Y) \le I(X{:}B) = S(\sum_x p_x \rho_x) - \sum_x p_x S(\rho_x)$. Define the *accessible information* $I_{\mathrm{acc}}(\sigma^{XB}) = \max_{\mathcal{M}} I(X:Y)$.

Show that the optimal measurement consists of rank-one elements and has no more than d^2 outcomes, where $d = \dim(B)$. Hint: the space of Hermitian operators on B is a vector space of size d^2.

Exercise 11.3 Quantum Data Processing Inequality

Consider two CPTP maps $\$_1$ and $\$_2$ acting on system Q. Call the initial state of Q ρ^Q, the output of the first map $\rho^{Q'} = \$(\rho^Q)$ and the output of the second map $\rho^{Q''} = \$_2 \circ \$_1(\rho^Q)$. Purifying the initial state with a system R and using the Stinespring dilations of the CPTP maps, we can regard this transformation as taking the pure state Ψ^{RQ} to $\Psi^{RQ'E_1}$ and then to $\Psi^{RQ''E_1E_2}$, where E_1 (E_2) is the environment of the first (second) map, so that E_1E_2 is the environment of the concatenated map $\$_2 \circ \$_1$. Now define the *coherent information* $I(A\rangle B) = -S(A|B)$. Show that

$$S(Q) \ge I(R\rangle Q') \ge I(R\rangle Q'').$$

Hint: use (strong) subadditivity.