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Quantum Information Theory
Solutions 7

HS 13
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Have a look at a nice and easy paper for bedtime reading: http://iopscience.iop.org/1464-4266/5/3/357
It is about photonical implementation of a simple POVM measurement.

Exercise 7.1 Generalized Measurement by Direct (Tensor) Product

Consider an apparatus whose purpose is to make an indirect measurement on a two-level system, A, by first coupling it to a three-level system, B, and then making a projective measurement on the latter. B is initially prepared in the state $|0\rangle$ and the two systems interact via the unitary U_{AB} as follows:

$$\begin{aligned} |0\rangle_A|0\rangle_B &\rightarrow \frac{1}{\sqrt{2}}\left(|0\rangle_A|1\rangle_B + |0\rangle_A|2\rangle_B\right) \\ |1\rangle_A|0\rangle_B &\rightarrow \frac{1}{\sqrt{6}}\left(2|1\rangle_A|0\rangle_B + |0\rangle_A|1\rangle_B - |0\rangle_A|2\rangle_B\right) \end{aligned}$$

1. *Calculate the measurement operators acting on A corresponding to a measurement on B in the canonical basis $|0\rangle, |1\rangle, |2\rangle$.*

   Name the output states $|\phi_{00}\rangle_{AB}$ and $|\phi_{01}\rangle_{AB}$, respectively. Although the specification of U is not complete, we have the pieces we need, and we can write $U_{AB} = \sum_{jk} |\phi_{jk}\rangle\langle jk|$ for some states $|\phi_{10}\rangle$ and $|\phi_{11}\rangle$. The measurement operators A_k are defined implicitly by

   $$U_{AB}|\psi\rangle_A|0\rangle_B = \sum_k (A_k)_A|\psi\rangle_A|k\rangle_B.$$

   Thus $A_k = {}_B\langle k|U_{AB}|0\rangle_B = \sum_j {}_B\langle k|\phi_{j0}\rangle_{AB}\langle j|_A$, which is an operator on system A, even though it might not look like it at first glance. We then find

   $$A_0 = \frac{2}{\sqrt{6}}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_1 = \frac{1}{\sqrt{6}}\begin{pmatrix} \sqrt{3} & 1 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} \sqrt{3} & -1 \\ 0 & 0 \end{pmatrix}.$$

2. *Calculate the corresponding POVM elements. What is their rank? Onto which states do they project?*

   The corresponding POVM elements are given by $E_j = A_j^\dagger A_j$:

   $$E_0 = \frac{2}{3}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad E_1 = \frac{1}{6}\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}, \quad E_2 = \frac{1}{6}\begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}.$$

   They are each rank one (which can be verified by calculating the determinant). The POVM elements project onto trine states $|1\rangle, (\sqrt{3}|0\rangle \pm |1\rangle)/2$.

3. *Suppose A is in the state $|\psi\rangle_A = \frac{1}{\sqrt{2}}(|0\rangle_A + |1\rangle_A)$. What is the state after a measurement, averaging over the measurement result?*

   The averaged post-measurement state is given by $\rho' = \sum_j A_j \rho A_j^\dagger$. In this case we have $\rho' = \mathrm{diag}(2/3, 1/3)$.

Exercise 7.2 Unambiguous State Discrimination

Suppose that Bob has a state ρ that can either be ρ_1 and ρ_2, but he does not know which one. Bob wants to guess which state he has, and he wants to never guess wrong. He can achieve that, if he is allowed to not make a guess at all based on result of his measurement.

1. *Bob's measurement surely has outcomes E_1 and E_2 corresponding to ρ_1 and ρ_2, respectively. Assuming the two states ρ_j are pure, $\rho_j = |\phi_j\rangle\langle\phi_j|$ for some $|\phi_j\rangle$, what is the general form of E_j such that $\Pr(E_j|\rho_k) = 0$ for $j \neq k$?*

   Since the two signal states are pure, they span a two-dimensional subspace and without loss of generality we can restrict the support of the POVM elements to this subspace—an effective qubit. Suppose E_j are rank-one operators $E_j = \alpha_k|\xi_j\rangle\langle\xi_j|$ (if they aren't, decompose them into a set of rank-one operators). Then we want to fulfill $0 = \Pr(E_j|\rho_k) = \alpha_k|\langle\xi_j|\phi_k\rangle|^2$, which can only work if $|\xi_j\rangle = |\phi_k^\perp\rangle$. That is, $|\xi_0\rangle$ is the state orthogonal to $|\phi_1\rangle$ and *vice versa*; the unambiguous measurement works by rejecting rather than confirming one of the two hypotheses. Thus $E_j = \alpha_j|\phi_k^\perp\rangle\langle\phi_k^\perp|$ for $j \neq k$ and some $0 \leq \alpha_k \leq 1$.

2. *Can these two elements alone make up a POVM? Is there generally an inconclusive result $E_?$?*

   Since $\langle\phi_1|\phi_2\rangle \neq 0$ in general, $\sum_{j=1}^{2} E_j \neq \mathbb{1}$, and therefore a third measurement element is needed. This outcome tells Bob nothing about which signal was sent, so it is an inconclusive result $E_?$.

3. *Assuming ρ_1 and ρ_2 are sent with equal probability, what is the optimal unambiguous measurement, i.e. the unambigous measurement with the smallest probability of an inconclusive result?*

   We know that a general unambiguous discrimination POVM has the form

$$E_0 = \alpha_0|\phi_1^\perp\rangle\langle\phi_1^\perp|, \qquad E_1 = \alpha_1|\phi_0^\perp\rangle\langle\phi_0^\perp|, \qquad E_? = \mathbb{1} - E_0 - E_1.$$

   The sum-to-unity constraint is enforced by the form of $E_?$ and $E_{0/1}$ are positive by construction, so the only outstanding constraint is that $E_?$ be positive. Symmetry between the signal states implies that $\alpha_0 = \alpha_1$, leaving

$$\mathbb{1} - \alpha(|\phi_0^\perp\rangle\langle\phi_0^\perp| + |\phi_1^\perp\rangle\langle\phi_1^\perp|) \geq 0.$$

   Thus we should choose the largest value of α consistent with this constraint. We can find a closed-form expression in terms of Bloch-sphere quantities. Let $|\phi_j\rangle$ have Bloch vector $\hat{n}_j$, meaning $|\phi_j^\perp\rangle$ has Bloch vector $-\hat{n}_j$. Then the constraint becomes

$$\mathbb{1} - \tfrac{1}{2}\alpha\left(\mathbb{1} - \hat{n}_1 \cdot \vec{\sigma} + \mathbb{1} - \hat{n}_0 \cdot \vec{\sigma}\right) = (1-\alpha)\mathbb{1} + \alpha(\hat{n}_0 + \hat{n}_1) \cdot \vec{\sigma} \geq 0.$$

   We know the eigenvalues of a general expression in terms of the Pauli operators and identity from the lecture on qubits, namely $\lambda_\pm = (1-\alpha) \pm \alpha|\hat{n}_0 + \hat{n}_1|$. Thus, the largest possible α is

$$\alpha = \frac{1}{1 + |\hat{n}_0 + \hat{n}_1|}.$$

   When the $|\phi_j\rangle$ are orthogonal, $\hat{n}_0 + \hat{n}_1 = 0$ and the unambiguous measurement goes over into the usual projection measurement.

Exercise 7.3 Decompositions of Density Matrices

Consider a mixed state ρ with two different pure state decompositions

$$\rho = \sum_{k=1}^{d} \lambda_k |k\rangle\langle k| = \sum_{\ell=1}^{n} p_\ell |\phi_\ell\rangle\langle\phi_\ell|,$$

the former being the eigendecomposition so that $\{|k\rangle\}$ is an orthonormal basis.

1. *Show that the probability vector $\vec{\lambda}$* majorizes *the probability vector $\vec{p}$, which means that there exists a doubly stochastic matrix T_{jk} such that $\vec{p} = T\vec{\lambda}$. The defining property of doubly stochastic, or bistochastic, matrices is that $\sum_k T_{jk} = \sum_j T_{jk} = 1$.*
   Hint: *Observe that for a unitary matrix U_{jk}, $T_{jk} = |U_{jk}|^2$ is doubly stochastic.*

By the HJW theorem we have $\sqrt{p_\ell}|\phi_\ell\rangle = \sum_k \sqrt{\lambda_k} U_{k\ell}|k\rangle$ for some unitary matrix $U_{k\ell}$. Taking the norm of each expression results in

$$p_\ell = \sum_k \lambda_k |U_{k\ell}|^2$$

since $|k\rangle$ is an orthonormal basis. Thus $\vec{\lambda}$ majorizes $\vec{p}$. Note that we cannot turn this argument around to say that $\vec{p}$ majorizes λ since starting from $\sqrt{\lambda_k}|k\rangle = \sum_\ell \sqrt{p_\ell} U^\dagger_{k\ell}|\phi_\ell\rangle$ we cannot easily compute the norm of the righthandside since the $|\phi_k\rangle$ are not orthogonal.

2. *The uniform probability vector $\vec{u} = (1/n, \dots, 1/n)$ is invariant under the action of an $n \times n$ doubly stochastic matrix. Is there an ensemble decomposition of ρ such that $p_\ell = 1/n$ for all ℓ?*
   Hint: *Try to show that $\vec{u}$ is majorized by any other probability distribution.*

   $\vec{u}$ is majorized by every other distribution $\vec{p}$ (of length less or equal to n) since we can use the doubly stochastic matrix $T_{jk} = 1/n$ for all j, k to produce $\vec{u} = T\vec{p}$. Therefore, to find a decomposition in which all the weights are identical, we need to find a unitary matrix whose entries all have the same magnitude, namely $1/\sqrt{n}$. One choice that exists in every dimension is the Fourier transform $F_{jk} = \frac{1}{\sqrt{n}}\omega^{jk}$, where $\omega = \exp(2\pi i/n)$. The vectors in the decomposition are therefore

$$|\phi_\ell\rangle = \sum_k \sqrt{\lambda_k}\omega^{k\ell}|k\rangle.$$

Exercise 7.4 Broken Measurement

Alice and Bob share a state $|\Psi\rangle_{AB}$, and Bob would like to perform a measurement described by projectors P_j on his part of the system, but unfortunately his measurement apparatus is broken. He can still perform arbitrary unitary operations, however. Meanwhile, Alice's measurement apparatus is in good working order. Show that there exist projectors P'_j and unitaries U_j and V_j so that

$$|\Psi_j\rangle = (\mathbb{1} \otimes P_j)|\Psi\rangle = (U_j \otimes V_j)\left(P'_j \otimes \mathbb{1}\right)|\Psi\rangle.$$

(Note that the state is unnormalized, so that it implicitly encodes the probability of outcome j.) Thus Alice can assist Bob by performing a related measurement herself, after which they can locally correct the state.
Hint: *Work in the Schmidt basis of $|\Psi\rangle$.*

Start with the Schmidt decomposition of $|\Psi\rangle_{AB}$:

$$|\Psi\rangle_{AB} = \sum_k \sqrt{p_k}|\alpha_k\rangle|\beta_k\rangle.$$

Bob's measurement projectors P_j can be expanded in his Schmidt basis as $P_j = \sum_{k\ell} c^j_{k\ell}|\beta_k\rangle\langle\beta_\ell|$. In order for Alice's measurement to replicate Bob's, the probabilities of the various outcomes must be identical, which is to say

$$\langle\Psi|(P_j)_B|\Psi\rangle_{AB} = \langle\Psi|(P'_j)_A|\Psi\rangle_{AB} \quad \Rightarrow \quad \sum_k p_k\langle\alpha_k|P'_j|\alpha_k\rangle = \sum_k p_k\langle\beta_k|P_j|\beta_k\rangle.$$

Thus Alice should choose $P'_j = \sum_{k\ell} c^j_{k\ell}|\alpha_k\rangle\langle\alpha_\ell|$. The post-measurement states when Alice or Bob measures are given by

$$|\Psi'_j\rangle = \sum_{k\ell} \sqrt{p_k} c^j_{k\ell}|\alpha_\ell\rangle|\beta_k\rangle \qquad \text{and} \qquad |\Psi_j\rangle = \sum_{k\ell}\sqrt{p_k}c^j_{k\ell}|\alpha_k\rangle|\beta_\ell\rangle,$$

respectively. Neither is in Schmidt form, but note that they are related by a simple swap operation $|\alpha_j\rangle_A|\beta_k\rangle_B \leftrightarrow |\alpha_k\rangle_A|\beta_j\rangle_B$, which is unitary; call it W_{AB} so that $|\Psi'_j\rangle = W|\Psi_j\rangle$. Now let $U'_j \otimes V'_j$ be unitary operators which transform $|\Psi_j\rangle$ to Schmidt form in the $|\alpha_j\rangle|\beta_k\rangle$ basis. That is, $(U'_j \otimes V'_j)|\Psi_j\rangle = \sum_k \sqrt{p^j_k}|\alpha_k\rangle|\beta_k\rangle$, and it follows that $W(U'_j \otimes V'_j)|\Psi_j\rangle = (U'_j \otimes V'_j)|\Psi_j\rangle$. Therefore $V'_j \otimes U'_j$ takes $|\Psi'_j\rangle$ to Schmidt form:

$$(V'_j \otimes U'_j)|\Psi'_j\rangle = WW^\dagger(V'_j \otimes U'_j)W|\Psi_j\rangle = W(U'_j \otimes V'_j)|\Psi_j\rangle = \sum_k \sqrt{p^j_k}|\alpha_k\rangle|\beta_k\rangle,$$

and thus

$$\begin{aligned} & (U'_j \otimes V'_j)|\Psi_j\rangle = (V'_j \otimes U'_j)|\Psi'_j\rangle \\ \Rightarrow\quad & (U'_j \otimes V'_j)(\mathbb{1} \otimes P_j)|\Psi\rangle = (V'_j \otimes U'_j)(P'_j \otimes \mathbb{1})|\Psi\rangle \\ \Rightarrow\quad & (\mathbb{1} \otimes P_j)|\Psi\rangle = (U'^\dagger_j V'_j \otimes V'^\dagger_j U'_j)(P'_j \otimes \mathbb{1})|\Psi\rangle. \end{aligned}$$