Quantum Information Theory
Solutions 10

Exercise 10.1 Teleportation Redux

(a) *Show that for the entangled state* $|\Phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ *and any unitary operator* U,

$$\left(U_A \otimes \overline{U}_B\right)|\Phi\rangle_{AB} = |\Phi\rangle_{AB},$$

where $^-$ *denotes complex conjugation in the* $|0\rangle, |1\rangle$ *basis.*

$$\begin{aligned}(U_A \otimes U_B^*)\,|\Phi\rangle_{AB} &= \frac{1}{\sqrt{2}}\sum_{jk\ell mt} U_{jk}U^*_{\ell m}\left(|j\rangle\langle k|_A \otimes |\ell\rangle\langle m|_B\right)|t,t\rangle_{AB}\\ &= \frac{1}{\sqrt{2}}\sum_{j\ell t} U_{jt}U^*_{\ell t}|j,\ell\rangle_{AB} = \frac{1}{\sqrt{2}}\sum_{j\ell}|j,\ell\rangle_{AB}\sum_t U_{jt}(U^\dagger)_{t\ell}\\ &= \frac{1}{\sqrt{2}}\sum_{j\ell}|j,\ell\rangle_{AB}\left[UU^\dagger\right]_{j\ell} = \frac{1}{\sqrt{2}}\sum_j |j,j\rangle_{AB} = |\Phi\rangle_{AB}\end{aligned}$$

(b) *Show that for any state* $|\psi\rangle$

$${}_A\langle\psi|\Phi\rangle_{AB} = \tfrac{1}{\sqrt{2}}|\psi^*\rangle_B.$$

$${}_A\langle\psi||\Phi\rangle_{AB} = \frac{1}{\sqrt{2}}\sum_{jk}\psi_j^*\langle j||k\rangle_A|k\rangle_B = \frac{1}{\sqrt{2}}\sum_k \psi_k^*|k\rangle_B = \frac{1}{\sqrt{2}}|\psi^*\rangle_B \tag{1}$$

(c) *Use the results of (a) and (b) to give a derivation of the teleportation protocol without resorting to components.*

Alice and Bob start out with the state $|\psi\rangle_{A'}|\Phi\rangle_{AB}$, where Alice holds systems A and A' and Bob B. When Alice measures $A'A$ in the Bell basis $|\Phi_j\rangle = (\mathbb{1}\otimes\sigma_j)\,|\Phi\rangle$, obtaining result j, the resulting state of $|\psi_j'\rangle_B$ Bob's system is

$$\begin{aligned}|\psi_j'\rangle_B &= {}_{A'A}\langle\Phi_j|\left(|\psi\rangle_{A'}|\Phi\rangle_{AB}\right) = {}_{A'A}\langle\Phi|\left(\mathbb{1}_{A'}\otimes(\sigma_j^\dagger)_A\otimes\mathbb{1}_B\right)|\psi\rangle_{A'}|\Phi\rangle_{AB}\\ &= {}_{A'A}\langle\Phi|\left(\mathbb{1}_{A'}\otimes\mathbb{1}_A\otimes(\sigma_j^*)_B\right)|\psi\rangle_{A'}|\Phi\rangle_{AB} = \tfrac{1}{\sqrt{2}}{}_A\langle\psi^*|\left(\mathbb{1}_A\otimes(\sigma_j^*)_B\right)|\Phi\rangle_{AB}\\ &= \tfrac{1}{2}(\sigma_j^*)_B|\psi\rangle_B = \tfrac{1}{2}(\sigma_j)_B|\psi\rangle_B.\end{aligned}$$

(The last equality follows since σ_x and σ_z have real entries, $\sigma_y^* = -\sigma_y$, and we don't care about overall phases.) Alice then tells Bob the j she obtained in the measurement (which takes two bits of communication), and then he applies σ_j^T to get $|\psi\rangle$.

(d) *What happens if Alice and Bob use the state* $(\mathbb{1}_A\otimes U_B)|\Phi\rangle_{AB}$ *for teleportation? Or if Alice measures in the basis* $\overline{U}_{A'}|\Phi_j\rangle_{A'A}$*?*

In the first case we have

$$|\psi_j'\rangle_B = {}_{A'A}\langle\Phi_j|\left(|\psi\rangle_{A'}U_B|\Phi\rangle_{AB}\right) = {}_{A'A}\langle\Phi|\left(\mathbb{1}_{A'}\otimes(\sigma_j^\dagger)_A\otimes U_B\right)|\psi\rangle_{A'}|\Phi\rangle_{AB}$$
$$= {}_{A'A}\langle\Phi|\left(\mathbb{1}_{A'}\otimes\mathbb{1}_A\otimes(U\sigma_j^*)_B\right)|\psi\rangle_{A'}|\Phi\rangle_{AB} = \tfrac{1}{2}(U\sigma_j^*)_B|\psi\rangle_B.$$

After Bob receives Alice's message and applies σ_j^T they end up with the state $|\psi_j''\rangle = (\sigma_j^T U\sigma_j^*)|\psi\rangle$. For the second case

$$|\psi_j'\rangle_B = {}_{A'A}\langle\Phi_j|U_{A'}\left(|\psi\rangle_{A'}U_B|\Phi\rangle_{AB}\right) = {}_{A'A}\langle\Phi_j|\left(|U\psi\rangle_{A'}U_B|\Phi\rangle_{AB}\right) = \tfrac{1}{2}(\sigma_j^*U)_B|\psi\rangle_B.$$

Now Bob's correction operation produces $|\psi''\rangle = U|\psi\rangle$. This is an important result, because it shows that it is possible to perform an arbitrary single-qubit operation solely by measuring an appropriately prepared state.

(e) *Instead of a single system state $|\psi\rangle_{A'}$, Alice has a bipartite state $|\psi\rangle_{A_1A_2}$. What happens if she performs the teleportation protocol on system A_2?*

Work with the Schmidt decomposition: $|\psi\rangle_{A_1A_2} = \sum_k \sqrt{p_k}|\alpha_k\rangle_{A_1}|\beta_k\rangle_{A_2}$. Then following the same calculation above we get

$$|\psi_j'\rangle_{A_1B} = {}_{A_2A}\langle\Phi_j|\left(|\psi\rangle_{A_1A_2}|\Phi\rangle_{AB}\right) = \sum_k \sqrt{p_k}\ {}_{A_2A}\langle\Phi_j|(|\alpha_k\rangle_{A_1}|\beta_k\rangle_{A_2}|\Phi\rangle_{AB})$$
$$= \sum_k \sqrt{p_k}|\alpha_k\rangle_{A_1}\ {}_{A_2A}\langle\Phi_j|(|\beta_k\rangle_{A_2}|\Phi\rangle_{AB}) = \tfrac{1}{2}\sum_k \sqrt{p_k}|\alpha_k\rangle_{A_1}(\sigma_j^*)_B|\beta_k\rangle_B$$
$$= \tfrac{1}{2}(\sigma_j^*)_B|\psi\rangle_{A_1B}.$$

Once again Bob can undo the σ_j^* on system B and thus teleportation can also faithfully transfer part of a larger, entangled system.

Exercise 10.2 Remote Copy

Alice and Bob would like to create the state $|\Psi\rangle_{AB} = a|00\rangle_{AB} + b|11\rangle_{AB}$ from Alice's state $|\psi\rangle_A = a|0\rangle_A + b|1\rangle_A$, a "copy" in the quantum-mechanical sense. Additionally, they share the canonical entangled state $|\Phi\rangle$. Can they create the desired state by performing only local operations (measurements and unitary operators), provided Alice can only send one *bit of classical information to Bob?*

By the solution to part (e) of the previous problem, Alice could create the copied state herself using the CNOT gate $U_{\text{CNOT}}|j,k\rangle = |j, j\oplus k\rangle$ and then teleport half of it to Bob. However, this would take two bits of communication. Suppose Alice copies $|\psi\rangle_A$ to her half of the maximally entangled state $|\Phi\rangle_{A'B}$. This results in

$$U_{\text{CNOT}}^{AA'}|\psi\rangle_A|\Phi\rangle_{A'B} = \tfrac{1}{\sqrt{2}}\left(a|000\rangle + a|011\rangle + b|110\rangle + b|101\rangle\right)_{AA'B}$$
$$= \tfrac{1}{\sqrt{2}}\left[(a|00\rangle + b|11\rangle)_{AB}|0\rangle_{A'} + (a|01\rangle + b|10\rangle)_{AB}|1\rangle_{A'}\right]$$
$$= \tfrac{1}{\sqrt{2}}\left(|\Psi\rangle_{AB}|0\rangle_{A'} + (\sigma_x)_B|\Psi\rangle_{AB}|1\rangle_{A'}\right).$$

As in teleportation, this creates the desired output state, up to the action of a Pauli operator on Bob's system which is indexed by an orthogonal state at Alice's end. By measuring system A' and telling Bob the result (using just one bit since there are only two outcomes) he can undo the Pauli operator to create $|\Psi\rangle_{AB}$.

Exercise 10.3 Quantum mutual information

Consider a composed system $A\otimes B\otimes C$ with a shared state ρ_{ABC}.

In a first step we ignore system C and consider only $A \otimes B$ (and the reduced state $\rho_{AB} = Tr_C(\rho_{ABC})$). One way of quantifying the correlations between A and B is to use the mutual information *between them, defined as*

$$I(A:B) = H(A) + H(B) - H(AB) \tag{2}$$
$$= H(A) - H(A|B). \tag{3}$$

If we have access to C, we can define a conditional version of the mutual information between A and B as

$$I(A:B|C) = H(A|C) + H(B|C) - H(AB|C) \tag{4}$$
$$= H(A|C) - H(A|BC). \tag{5}$$

(a) *Assume a system formed by two qubits A and B that share a state ρ_{AB}. Consider bases $\{|0\rangle_A, |1\rangle_A\}$ and $\{|0\rangle_B, |1\rangle_B\}$ for the subsystems of each qubit.*

1. *Check that the mutual information of the fully entangled state , $|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, is maximal.*
   The global state is pure and the reduced states on A and B are both fully mixed, $\rho_A = \rho_B = \mathbb{1}/2$, so we have
   $$H(AB) = 0, \quad H(A) = H(B) = 1 \quad \Rightarrow \quad I(A:B) = 2,$$
   which is maximal, because the entropy of a single qubit is at most $\log|\mathcal{H}_A| = 1$, as we saw in exercise 11.2, and the entropy of the joint state is always non negative.
2. *See that for classically correlated states, $\rho_{AB} = p|0\rangle\langle 0|0_A \otimes \sigma_B^0 + (1-p)|1\rangle\langle 1|1_A \otimes \sigma_B^1$ (where $0 \leq p \leq 1$), the mutual information cannot be greater than one.*
   We can rewrite the mutual information as
   $$I(A:B) = \underbrace{H(A)}_{\leq 1} - \underbrace{H(A|B)}_{\geq 0^{(*)}} \leq 1$$
   where $^{(*)}$ comes from exercise 11.1.b)3.

(b) *Consider the so-called* cat state *shared by four qubits, $A \otimes B \otimes C \otimes D$, that is defined as*

$$|☺\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle). \tag{6}$$

Check how the mutual information between qubits A and B changes with the knowledge of the remaining qubits, namely:

1. $I(A:B) = 1$.
2. $I(A:B|C) = 0$.
3. $I(A:B|CD) = 1$.

The reduced states of the system for k qubits (which are independent of the qubits traced out) have entropies denoted by h_k, given as follows:

$$\rho_4 = |☺\rangle\langle ☺|☺ \qquad \Rightarrow h_4 = 0,$$
$$\rho_3 = \frac{1}{2}(|000\rangle\langle 000|000 + |111\rangle\langle 111|111) \qquad \Rightarrow h_3 = 1,$$
$$\rho_2 = \frac{1}{2}(|00\rangle\langle 00|00 + |11\rangle\langle 11|11) \qquad \Rightarrow h_2 = 1,$$
$$\rho_1 = \frac{1}{2}(|0\rangle\langle 0|0 + |1\rangle\langle 1|1) \qquad \Rightarrow h_1 = 1.$$

The mutual information between A and B given the knowledge of other qubits comes

$$
\begin{aligned}
I(A:B) &= H(A) + H(B) - H(AB) \\
&= h_1 + h_1 - h_2 = 1, \\
I(A:B|C) &= H(A|C) + H(B|C) - H(AB|C) \\
&= H(AC) - H(C) + H(BC) - H(C) - H(ABC) + H(C) \\
&= h_2 - h_1 + h_2 - h_1 - h_3 + h_1 = 0, \\
I(A:B|CD) &= H(A|CD) + H(B|CD) - H(AB|CD) \\
&= H(ACD) - H(CD) + H(BCD) - H(CD) - H(ABCD) + H(CD) \\
&= h_3 - h_2 + h_3 - h_2 - h_4 + h_2 = 1.
\end{aligned}
$$