Exercise 1. ***Classical Field Symmetries***

Consider a Lagrangian decomposed in kinetic, mass and interaction parts as follows

$$\mathcal{L} = \mathcal{L}_{kin} + \mathcal{L}_{mass} + \mathcal{L}_{int}.$$

Each of these terms have generally different symmetries. The only mandatory term is $\mathcal{L}_{kin}$ and hence it defines the largest symmetry group of the full Lagrangian $\mathcal{L}$. The mass term $\mathcal{L}_{mass}$ and interaction term $\mathcal{L}_{int}$ have generally less symmetries and break this group to one of its subgroup. It will be often the case that such terms can be treated as perturbations.

We first look at n real scalar fields ϕ_i, $i = 1, \dots, n$, with kinetic term

$$\mathcal{L}_{kin} = \frac{1}{2}\partial_\mu \phi^T \partial^\mu \phi, \qquad \text{where} \quad \phi = (\phi_1, \dots, \phi_n)^T.$$

(a) What is the largest (global) symmetry of $\mathcal{L}_{kin}$? What is the dimension of the group and what are its generators?

(b) Compute the associated Noether current, assuming $\mathcal{L}_{mass} = \mathcal{L}_{int} = 0$.

(c) Let us introduce a real symmetric positive semi-definite *mass matrix* M and add a general mass term as

$$\mathcal{L}_{mass} = -\frac{1}{2}\phi^T M \phi = -\frac{1}{2}\phi_i M_{ij} \phi_j.$$

Such a term will in general break the symmetry group of $\mathcal{L}_{kin}$ to one of its subgroups. What is the symmetry group of $\mathcal{L}_{mass}$ and how does it relate to the eigenvalues of M?

(d) We can introduce a $n \times n$ matrix A and write a more general kinetic term as

$$\mathcal{L}_{kin} = \frac{1}{2}\partial_\mu \phi^T A \, \partial^\mu \phi = \frac{1}{2}\partial_\mu \phi_i A_{ij} \partial^\mu \phi_j.$$

What are the physical requirements on the eigenvalues of A? What is then the largest symmetry?

We now look at n fermion fields with kinetic term

$$\mathcal{L}_{kin} = i\bar{\psi}\partial\!\!\!/\psi.$$

(e) One obvious symmetry of this term is $\psi \to U\psi$ where U is a $n \times n$ unitary matrix. What are the dimension and generators of this group? Compute the corresponding Noether currents, assuming $\mathcal{L}_{mass} = \mathcal{L}_{int} = 0$.

(f) A less obvious symmetry of this kinetic term is the *chiral symmetry*. It is given by

$$\psi_L \to U_L \psi_L, \quad \psi_R \to U_R \psi_R$$

where U_L and U_R are $n \times n$ unitary matrices and the left and right components of ψ are

$$\psi_L = P_+ \psi, \quad \psi_R = P_- \psi, \qquad \text{where} \quad P_\pm = (1 \pm \gamma_5)/2.$$

Show that this is indeed a symmetry of $\mathcal{L}_{kin}$. Compute its Noether current, assuming $\mathcal{L}_{mass} = \mathcal{L}_{int} = 0$.

(g) Is the typical Dirac fermion mass term $-m\bar{\psi}\psi$ invariant under chiral symmetry?

Exercise 2. ***Charge Algebra***

Consider the Lagrangian

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$$

where $\mathcal{L}_0$ is invariant under some symmetry group G while $\mathcal{L}_1$ is not. We can still define the current associated to G by

$$J_a^\mu = -i \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} T^a \phi.$$

(a) Compute $\partial_\mu J_a^\mu$ and show that the charges $Q^a(t)$ are not conserved, i.e. $\partial_t Q^a(t) \neq 0$.

(b) Nevertheless, the charges still generate the transformation. Show that their equal time commutation relation is

$$[Q^a(t), Q^b(t)] = i f^{abc} Q^c,$$

where f^{abc} are the structure constants of G, i.e. $[T^a, T^b] = i f^{abc} T^c$.

Hint. Recall that the cannonical momentum is defined as

$$\pi(x,t) = \frac{\delta \mathcal{L}}{\delta(\partial_0 \phi)}, \qquad \text{with} \quad [\pi(x,t), \phi(y,t)] = -i\delta(x-y).$$