Exercise 1. ***Low-energy pion scattering***

In this exercise we look at pion scattering in the non-linear σ-model. The effective Lagrangian for the pion sector is given by

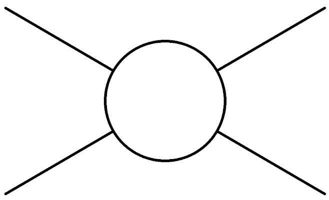
$$\mathcal{L}_\pi = \frac{F^2}{2}\vec{D}_\mu \cdot \vec{D}^\mu - \frac{c_4}{4}(\vec{D}_\mu \cdot \vec{D}^\mu)(\vec{D}_\nu \cdot \vec{D}^\nu) - \frac{c_4'}{4}(\vec{D}_\mu \cdot \vec{D}_\nu)(\vec{D}^\mu \cdot \vec{D}^\nu) - \ldots \tag{1}$$

The dots indicate higher order effective operators and $\vec{D}^\mu$ is the covariant derivative given by

$$\vec{D}^\mu = \frac{\partial^\mu \vec{\pi}}{1 + \vec{\pi}^2/F^2},$$

where the constant F is the pion decay amplitude and $\vec{\pi} = (\pi_1, \ldots, \pi_4)$ are the pion fields.

We look here at the 4–pion scattering $\pi_a(p_1)\pi_b(p_2) \to \pi_c(p_3)\pi_d(p_4)$ in the low–energy limit, i.e the limit where the energy of the pions is much lower than F, and want to compute the corresponding scattering amplitude as a series in $1/F^2$.



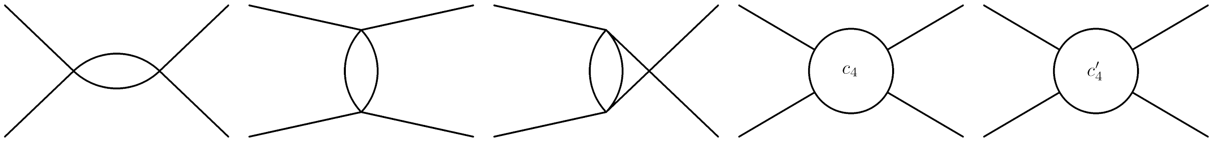
(a) Show that in the limit where $F \to \infty$ the part of $\mathcal{L}_\pi$ relevant to 4–pion scattering is given by

$$\begin{aligned}\mathcal{L}_\pi = &\frac{1}{2}(\partial_\mu \vec{\pi} \cdot \partial^\mu \vec{\pi}) - \frac{1}{F^2}(\partial_\mu \vec{\pi} \cdot \partial^\mu \vec{\pi})\vec{\pi}^2 + \frac{1}{F^4}(\partial_\mu \vec{\pi} \cdot \partial^\mu \vec{\pi})(\vec{\pi}^2)^2 \\ &- \frac{c_4}{4F^4}(\partial_\mu \vec{\pi} \cdot \partial^\mu \vec{\pi})(\partial_\nu \vec{\pi} \cdot \partial^\nu \vec{\pi}) - \frac{c_4'}{4F^4}(\partial_\mu \vec{\pi} \cdot \partial_\nu \vec{\pi})(\partial^\mu \vec{\pi} \cdot \partial^\nu \vec{\pi}) \\ &+ \text{higher point interactions} + \mathcal{O}(F^{-6}).\end{aligned}$$

(b) The leading contribution to the amplitude is only given by the vertex $(\partial_\mu \vec{\pi} \cdot \partial^\mu \vec{\pi})\vec{\pi}^2/F^2$. By obtaining the Feynman rules, show that this contribution is

$$\frac{4}{F^2}\left(\delta_{ab}\delta_{cd}(p_1 \cdot p_2 + p_3 \cdot p_4) - \delta_{ac}\delta_{bd}(p_1 \cdot p_3 + p_2 \cdot p_4) - \delta_{ad}\delta_{bc}(p_1 \cdot p_4 + p_2 \cdot p_3)\right). \tag{2}$$

At order F^{-4}, we get loop contributions arising from the vertex (2) and contributions coming from the terms proportional to c_4 and c_4'. The corresponding diagrams read



(c) The loop integrals need to be regularised since they are ultra-violet divergent. Using a cutoff Λ show that the bubble intergal reads

$$\int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{k^2(k+P)^2} = \frac{1}{16\pi^2}\left(1 - \log P^2 + \log \Lambda^2\right),$$

in the $\Lambda \to \infty$ limit.

(d) Show that the diagrams sum up to

$$-\frac{\delta_{ab}\delta_{cd}}{F^4}\left(\frac{s^2\log(s)}{2\pi^2} - \frac{(u^2-s^2+3t^2)\log(t)}{12\pi^2} - \frac{(t^2-s^2+3u^2)\log(u)}{12\pi^2} + \frac{(s^2+t^2+u^2)\log\Lambda^2}{3\pi^2} - \frac{1}{2}c_4 s^2 - \frac{1}{4}c_4'(t^2+u^2)\right) + \text{crossed terms},$$

where 'crossed terms' denotes terms given by interchanging the pions $2 \leftrightarrow 3$ and $2 \leftrightarrow 4$, and s, t, and u are the Mandelstam variables

$$s = (p_1+p_2)^2, \qquad t = (p_1-p_3)^2, \qquad u = (p_1-p_4)^2.$$

Finally, note that the ultra-violet divergences can be absorbed by renormalization of the constants

$$c_{4R} = c_4 - \frac{2}{3\pi^2}\log\left(\frac{\Lambda^2}{\mu^2}\right), \quad c_{4R}' = c_4' - \frac{4}{3\pi^2}\log\left(\frac{\Lambda^2}{\mu^2}\right).$$

Hence, we see that even if we are working with an effective Lagrangian, it is possible to absorb all divergences by including higher dimensional operators in (1), order by order.