Exercise 1. ***Adding flavour***

(a) What are the masses and electro-magnetic charges of the Π, K, and η mesons?

The previously discussed $SU_L(2) \otimes SU_R(2)$ symmetric QCD Lagrangian with two quark flavors can be extended to the three flavor case as the strange quark is still comparably light. We can combine the three quark flavors into one triplet with mass m and electric charge Q.

$$q = \begin{pmatrix} u \\ d \\ s \end{pmatrix}, \qquad m = \begin{pmatrix} 3MeV \\ 6MeV \\ 150MeV \end{pmatrix}, \qquad Q = \begin{pmatrix} \frac{2}{3} \\ \frac{-1}{3} \\ \frac{-1}{3} \end{pmatrix}. \tag{1}$$

The QCD Lagrangian without any mass term has now a $SU_L(3) \otimes SU_R(3)$ symmetry. In analogy to the 2-flavor case we consider an effective description of the Goldstone bosons that would arise from spontaneously breaking $SU_L(3) \otimes SU_R(3) \to SU_V(3)$.

(b) How many Goldstone bosons do you expect?

Such a description is given by the non linear sigma model.

$$\mathcal{L} = \frac{F^2}{16} Tr\left[\partial_\mu U \partial^\mu U\right],$$

with

$$U = e^{i\frac{2\sqrt{2}}{F}B}, \qquad B = \begin{pmatrix} \frac{1}{\sqrt{2}}\Pi_0 + \frac{1}{\sqrt{6}}\eta^0 & \Pi^+ & K^+ \\ \Pi^- & -\frac{1}{\sqrt{2}}\Pi^0 + \frac{1}{\sqrt{6}}\eta^0 & K^0 \\ \bar{K}^- & \bar{K}^0 & -\sqrt{\frac{2}{3}}\eta^0 \end{pmatrix}.$$

You could have chosen $B = \sum_i \xi_i \lambda_i$ for general $\xi_i(x)$ and λ being the Gell-Mann matrices. Applying a symmetry transformation we find $U \to g_L U g_R^\dagger$.

(c) Check the assignment of electric charges in our matrix B by applying an $U_{em}(1)$ electro-magnetic transformation.

$$U \to e^{i\alpha Q} U e^{-i\alpha Q}, \qquad Q = \begin{pmatrix} q_u & 0 & 0 \\ 0 & q_d & 0 \\ 0 & 0 & q_s \end{pmatrix}.$$

Regard an infinitesimal transformation. What do you conclude?

(d) Demanding that the fields we denoted by Π should indeed correspond to the pions we derived from the 2-flavor case, what do you learn about F?

The $SU_L(3) \otimes SU_R(3)$ symmetry can be explicitly broken by introducing a new field $M(x)$ without a dynamic degree of freedom. Such fields are called spurions. We introduce it such that the field itself has the same symmetry as our non-linear sigma model, $M \to g_L M g_R^\dagger$. We choose

M to be real and we break the symmetry by assigning a non-vanishing vacuum expectation value.

$$\langle M \rangle_0 = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix} .$$

Weinberg's principle of constructing Lagrangians states that you have to write down every operator that is allowed by the symmetry and particle content of your theory.

(e) Convince yourself that the term that satisfies the symmetry at leading power in the new field M is given by

$$\mathcal{L}_m = -\phi \Lambda_{QCD}^3 Tr\left[M(U + U^\dagger) \right] .$$

ϕ is an unknown parameter.

(f) Calculate the masses arising for the Goldstone bosons.

(g) Check the following relation numerically and analytically:

$$3m_\eta^2 + 2m_{\Pi^\pm}^2 - m_{\Pi^0}^2 = 2m_{K^\pm}^2 + 2m_{K^0}^2 .$$

(h) Compute the numeric rations of the masses of the mesons from eq. (1) and compare them to the values you found at the first exercise.