Exercise 1. ***Higgs mechanism***

We consider the simplest case of a $U(1)$ gauge theory with complex scalar field $\phi(x)$

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + (D_\mu\phi)^*(D^\mu\phi) - m^2|\phi|^2 - \lambda|\phi|^4,$$

where the covariant derivative is $D_\mu = \partial_\mu + igA_\mu$. For $m^2 < 0$ the minimum of the potential occurs at

$$|\phi| = v = \sqrt{-m^2/\lambda},$$

such that we have spontaneous breakdown of the local symmetry.

(a) By writing $\phi(x) = (v + \rho(x))\,\mathrm{e}^{i\frac{\eta(x)}{v}}$ with $\langle\rho\rangle = \langle\eta\rangle = 0$ show that the A_μ independent part of the Lagrangian becomes

$$\mathcal{L}_\phi = \frac{1}{2}\partial_\mu\rho\partial^\mu\rho + \frac{1}{2}\partial_\mu\eta\partial^\mu\eta - v^2\lambda\rho^2 + \ldots,$$

where the dots indicate higher order terms. What field would you identify as the Goldstone boson? Thinking about the A_μ dependent terms, is such an interpretation valid?

(b) Although the situation looks the same as in the case of the breaking of a continuous global symmetry, the particle content of the theory is in fact very different. To make the spectrum of the theory apparent it is relevant to work in the so called *unitary gauge*: Use the gauge freedom to perform the transformation

$$\phi(x) \to \mathrm{e}^{-i\frac{\eta(x)}{v}}\phi(x).$$

Obtain the corresponding transformation for $A_\mu(x)$ and show that the full Lagrangian becomes

$$\mathcal{L} \to -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\partial_\mu\rho\partial^\mu\rho + \frac{1}{2}g^2(\rho+v)^2A_\mu A^\mu - \frac{1}{4}\lambda(\rho^2 + 2\rho v).$$

What is the particle content of the theory? Is the Goldstone boson still present?

The unitary gauge makes the particle content of a theory apparent but is not manifestly renormalizable. Hence it is useful to consider different gauges.

(c) Let us first look at the covariant gauge with gauge fixing term $-(\partial_\mu A^\mu)^2/2a$. Writing $\phi(x) = (\varphi(x) - v + i\chi(x))/\sqrt{2}$, show that the Lagrangian becomes

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - \lambda v^2\varphi^2 - \frac{1}{2a}(\partial_\mu A^\mu)^2 \\ &+ \frac{1}{2}g^2v^2A_\mu A^\mu + gvA_\mu\partial^\mu\chi \\ &+ \ldots,\end{aligned} \tag{1}$$

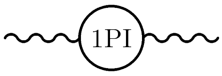
where the dots indicate interaction terms. The coupling $gvA_\mu\partial^\mu\chi$ would lead to a system of coupled differential equations for the A_μ and χ propagators. Instead we will take the first line of $\mathcal{L}$ as the free Lagrangian with propagators (in the Landau gauge $a \to 0$)

$$
\begin{aligned}
A_\mu: &\qquad \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\frac{-i}{p^2} \equiv d_{\mu\nu}\frac{-i}{p^2}, \\
\varphi: &\qquad \frac{i}{p^2 - 2\lambda v^2}, \\
\chi: &\qquad \frac{i}{p^2}.
\end{aligned}
$$

and expand the second line. At first sight it seems that the gauge boson A_μ remains massless! In fact, the full gauge boson propagator can be obtained by resuming the Dyson series

$$\Delta^{\mu\nu}(p) = \frac{id^{\mu\nu}}{p^2} + \frac{-id^{\mu\rho}}{p^2} i\Pi_{\rho\sigma}(p^2)p^2\frac{-id^{\sigma\nu}}{p^2} + \ldots = \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\frac{-i}{p^2}\frac{1}{1-\Pi(p^2)},$$

where $\Pi_{\mu\nu}(p^2) = i(g_{\mu\nu}p^2 - p_\mu p_\nu)\Pi(p^2)$ is the sum of all 1PI diagrams

$\Pi^{\mu\nu}(p^2) =$ μ  ν

Show that the second line of (1) contributes as $\Pi(p^2) = g^2v^2/p^2 + \mathcal{O}(g^4)$ such that we obtain

$$\Delta_{\mu\nu}(p) = -i\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\frac{1}{p^2 - g^2v^2}.$$

What is the interpretation of such a propagator?

(d) We can get rid of the problematic coupling $gvA_\mu\partial^\mu\chi$ by using the R_ξ gauge with gauge fixing term

$$-\frac{1}{2\xi}(\partial^\mu A_\mu - \xi gv\xi)^2.$$

Show that by adding such a gauge fixing instead of $-(\partial_\mu A^\mu)^2/2a$ the the Lagrangian (1) become

$$
\begin{aligned}
\mathcal{L} = &-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - \lambda v^2\varphi^2 \\
&+ \frac{1}{2}g^2v^2A_\mu A^\mu - \frac{1}{2}\xi g^2v^2\chi^2 - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \\
&+ \ldots.
\end{aligned}
\tag{2}
$$

Can you read out the propagators for the fields A_μ, φ and χ ?

Although the R_ξ gauges introduce fake poles at $p^2 = \xi g^2v^2$, those should cancel in any physically relevant quantity, because of gauge invariance.