# General relativity. Problem set 2.

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*HS 14* *Due:* Tue, September 30, 2014

## 1. Jacobi identity

i) Let $X, Y, Z$ be vector fields on a manifold $M$. Verify that the commutator satisfies the Jacobi identity:

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0\ .$$

ii) Let $Y_1, \ldots, Y_n$ be vector fields on an $n$-dimensional manifold $M$ such that at each $p \in M$ they form a basis of the tangent space $T_p(M)$. Then, at each point, we may expand each commutator $[Y_\alpha, Y_\beta]$ in this basis, thereby defining functions $C^\gamma{}_{\alpha\beta} = -C^\gamma{}_{\beta\alpha}$ by

$$[Y_\alpha, Y_\beta] = C^\gamma{}_{\alpha\beta} Y_\gamma\ . \tag{1}$$

Use the Jacobi identity to derive an equation satisfied by $C^\gamma{}_{\alpha\beta}$.

## 2. About the Lie derivative

In class the Lie derivative $L_X R$ of a tensor field $R$ was defined in 'absolute terms', i.e. without reference to charts or to components. It was then shown that, for the case of $R$ being of type $\binom{1}{1}$, the components of $L_X R$ are

$$(L_X R)^i{}_j = R^i{}_{j,k} X^k - R^k{}_j X^i{}_{,k} + R^i{}_k X^k{}_{,j}\ . \tag{2}$$

By contrast adopt here the point of view according to which $R$ is simply given by its components $R^i{}_j$ together with the transformation law

$$\bar{R}^\alpha{}_\beta = R^i{}_j \frac{\partial \bar{x}^\alpha}{\partial x^i} \frac{\partial x^j}{\partial \bar{x}^\beta} \tag{3}$$

under any change $x \mapsto \bar{x}$ of coordinates. Then take (2) as a definition of $L_X R$. Make sure it is well-defined by showing that $(L_X R)^i{}_j$ also obeys (3).

*Hint:* Find first the transformation law of $R^i{}_{j,k}$ (it is not that of a tensor). Be ready for a lengthy computation.