General relativity. Problem set 8.

HS 14 *Due:* Tue, November 11, 2014

1. Charged dust

Consider a charged dust consisting of particles of mass m and electric charge e.

i) Derive the equations of motion for $\rho(x)$ (mass density in the local rest frame) and $u^\mu(x)$ (4-velocity) in an electromagnetic field $F_{\mu\nu}(x)$. Show that the 4-current $j^\mu(x)$ satisfies

$$j^\mu{}_{;\mu} = 0$$

without making use of the Maxwell equations.

Hint: The equation of motion of a charged particle is (4.15)[1].

ii) Let $T^{\mu\nu}_{\text{em}}$, $T^{\mu\nu}_{\text{d}}$ be the energy-momentum tensors of the electromagnetic field, resp. of the charged dust. Show that

$$(T^{\mu\nu}_{\text{em}} + T^{\mu\nu}_{\text{d}})_{;\nu} = 0 \ .$$

Hint: In special relativity, $T^{\mu\nu}_{\text{em},\nu} = -c^{-1} F^{\mu\nu} j_\nu$.

2. On conservation laws

In special relativity the fact that currents and energy-momentum tensors are divergence-free, $j^\nu{}_{,\nu} = 0$, $T^{\mu\nu}{}_{,\nu} = 0$, implies akin integral formulations as conservation laws: The total charge, resp. energy-momentum vector

$$Q(t) = \int_{x^0 = ct} j^0 d^3x \ , \qquad P^\mu(t) = \int_{x^0 = ct} T^{\mu 0} d^3x$$

are independent of time t, assuming fields decaying at spatial infinity (see Electrodynamics). Not so in general relativity. Among the equations

$$j^\nu{}_{;\nu} = 0 \ , \qquad T^{\mu\nu}{}_{;\nu} = 0$$

only the first one admits such a formulation: The charge

$$Q(\Sigma) = \int_\Sigma (j, n) \sqrt{-g_\Sigma} \, d^3x \tag{1}$$

is independent of the spacelike 3-surface $\Sigma \subset M$ extending to spatial infinity (see below for notation).

i) Derive (1) using Gauss' theorem in the form ($D \subset M$ a bounded domain, W a vector field)

$$\int_D W^\nu{}_{;\nu} \sqrt{-g} \, d^4x = \int_{\partial D} (W, n) \sqrt{\mp g_{\partial D}} \, d^3x \ , \tag{2}$$

where $(\cdot, \cdot) = g(\cdot, \cdot)$ is the spacetime metric; $g(x) = \det(g_{\mu\nu}(x))$, and likewise for the induced metric $g_{\partial D}$: $(X, Y)_{\partial D} = (X, Y)$ for $X, Y \in T_p(\partial D)$, ($p \in \partial D$); and n is the

[1] equation numbers as in notes presently posted on the website

outward unit normal: $(n,n) = \pm 1$, $(n,X) = 0$. (The upper sign applies at p where ∂D is spacelike.)

Hint: Show $Q(\Sigma_1) = Q(\Sigma_2)$ for two spacelike surfaces Σ_i, $(i = 1,2)$.

ii) Prove (2) using

$$W^\nu{}_{;\nu}\sqrt{-g} = (W^\nu\sqrt{-g})_{,\nu} \ , \tag{3}$$

cf. (5.24), and Gauss' theorem in its basic form ($D \subset \mathbb{R}^4$)

$$\int_D X^\nu{}_{,\nu}\, d^4x = \int_{\partial D} X^\nu\, do_\nu$$

with do_ν the (coordinate) normal surface element.

Hint: Use local coordinates so that

$$g = \left(\begin{array}{c|c} g_{00} & 0 \\ \hline 0 & g_{\partial D} \end{array}\right) , \qquad (p \in \partial D).$$

iii) Show that (3) goes wrong when trying to extend the procedure to tensors,

$$T^{\mu\nu}{}_{;\nu}\sqrt{-g} \neq (T^{\mu\nu}\sqrt{-g})_{,\nu} \ .$$

Remark: In special cases (isolated systems, symmetric spacetimes) one may introduce a total energy and/or momentum.

3. Bound on the cosmological constant

Consider the modification (5.17) of the field equations by the cosmological constant Λ.

i) How are they modified when written in terms of the Ricci tensor, cf. (5.11)? How is the Poisson equation (5.15)?

ii) Show that the solution $\varphi = -G_0M/r$ for the gravitational potential generated by a point mass M is modified to

$$\varphi(\vec{x}) = -\frac{G_0M}{r} - \frac{1}{6}\Lambda c^2 r^2 \ .$$

iii) How small has Λ to be, so that its influence on the dynamics of the solar system is negligible?

Hint: Orbital radius of Pluto $r \cong 6 \cdot 10^{12}\,\mathrm{m}$, mass of the Sun $M \cong 2 \cdot 10^{30}\,\mathrm{kg}$, $G_0 \cong 6.7 \cdot 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\mathrm{s}^{-2}$, $c \cong 3 \cdot 10^8\,\mathrm{m\,s}^{-1}$.