General relativity. Problem set 10.

HS 14 *Due:* Tue, November 25, 2014

1. Radiation dominated universe

The model of a homogeneous and isotropic universe (Friedmann model) was treated in detail in the case that $T^{\mu\nu}$ is the energy-momentum tensor of dust. Consider now a homogeneous isotropic radiation field ($c = 1$)

$$T^{\mu\nu} = \frac{\varepsilon}{3}(4u^\mu u^\nu - g^{\mu\nu})$$

with $\varepsilon(t)$ the energy density and u the velocity w.r.t. which isotropy applies, i.e. $u^\mu = \delta^\mu{}_0$ in comoving coordinates. (Recall: This corresponds to a perfect fluid with $p = \varepsilon/3$ and follows from $T^\mu{}_\mu = 0$ or from exercise 7.1.) Consider the Friedmann equations for $a(t)$ in the form (6.13, 6.14).

i) Conclude that $\varepsilon a^4/3$ is conserved. Interpret the result in terms of photons undergoing cosmological redshift.

Hint: The conservation is essentially shown in the notes.

ii) Let $\Lambda = 0$ (radiation dominated universe, RD). Find the solutions $a(t)$.

Hint: Introduce instead of t the new coordinate

$$\eta = \int_0^t \frac{dt'}{a(t')} \, .$$

This is the same coordinate as used in the corresponding solution (6.25, 6.26) for dust (matter dominated universe, MD). The solution should be likewise given in parametric form $(a(\eta), t(\eta))$.

2. The causal structure of the Friedmann models

Using the above conformal time η, the metric of the Friedmann models reads, cf. (6.30)

$$g = a^2(\eta)\big(d\eta^2 - (d\chi^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2))\big) \, ,$$

where $r = r(\chi)$ was defined in (6.4).

i) Compute the range $\eta \in [0, \eta_0]$ in the cases MD and RD. The endpoints correspond to the Big Bang and, if $\eta_0 < +\infty$, to the Big Crunch.

ii) Let $k = 1$, i.e. space be a 3-sphere. Consider null geodesics beginning at $\chi = 0$ and represent them in a diagram $(\chi, \eta) \in [0, \pi] \times [0, \eta_0]$. Is it possible for an observer to send forward a light signal that will later hit him on his back?