Quantum Field Theory I

Problem Sets

ETH Zurich, HS14

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Contents

1.1. Classical particle in an electromagnetic field

Consider the classical Lagrangian density of a particle of mass m and charge q, moving in an electromagnetic field, specified by the electric potential $\phi(\vec{x},t)$ and the magnetic vector potential $\vec{A}(\vec{x},t)$:

$$\mathcal{L} = \tfrac{1}{2}m\dot{\vec{x}}^2 + q\vec{A}\cdot\dot{\vec{x}} - q\phi. \tag{1.1}$$

Determine the following quantities, and compare the results to those for a free particle:

a) the canonical momentum p_i conjugate to the coordinate x^i;

b) the equations of motion corresponding to the Lagrangian density;

c) the Hamiltonian of the system.

1.2. Stress-energy tensor

Consider the variational principle:

$$\delta S = \delta \int d^4x\, \mathcal{L}(\phi,\rho) = 0. \tag{1.2}$$

The Lagrangian density $\mathcal{L}$ is a function of the classical field $\phi(x)$ and its derivative $\rho_\mu(x) = \partial_\mu\phi(x)$. Note that $\mathcal{L}$ does not depend directly on the spacetime coordinate x^μ, but only indirectly through ϕ and ρ. Show that the conserved Noether current associated to infinitesimal spacetime translations

$$\delta x^\mu = \epsilon^\mu \tag{1.3}$$

is the stress-energy tensor $T^{\mu\nu}$ given by

$$T^{\mu\nu} = -\frac{\partial\mathcal{L}}{\partial\rho_\mu}\,\rho^\nu + \eta^{\mu\nu}\mathcal{L}. \tag{1.4}$$

Remind yourself how a general function $f(x^\mu)$ of the spacetime coordinates will transform under an infinitesimal translation. *Note:* $\eta^{\mu\nu} = \mathrm{diag}(-1,+1,+1,+1)$ is the metric tensor.

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1.3. Coherent quantum oscillator

Consider the Hamiltonian of a quantum harmonic oscillator:

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} \,. \tag{1.5}$$

a) Introduce ladder operators to diagonalise the Hamiltonian.

b) Calculate the expectation values of the number operator $N \sim a^\dagger a$ as well as of the x and p operator in a general number state $|n\rangle$.

c) Calculate the variances Δx, Δp and ΔN in the same state $|n\rangle$ and use them to determine the Heisenberg uncertainty of $|n\rangle$.

d) Show that the coherent state

$$|\alpha\rangle = e^{\alpha p}|0\rangle \tag{1.6}$$

is an eigenstate of the annihilation operator you defined in part a).

e) Calculate the time-dependent expectation values of x, p and N,

$$\langle\alpha|x(t)|\alpha\rangle, \qquad \langle\alpha|p(t)|\alpha\rangle, \qquad \langle\alpha|N(t)|\alpha\rangle, \tag{1.7}$$

as well as the corresponding variances to determine the uncertainty of the state $|\alpha\rangle$. Compare your result with the result obtained in part c).

1.4. Relativistic point particle

The action of a relativistic point particle is given by

$$S = -\alpha \int_{\mathcal{P}} ds \tag{1.8}$$

with the relativistic line element

$$ds^2 = -\eta_{\mu\nu} dx^\mu dx^\nu = dt^2 - dx^2 - dy^2 - dz^2 \tag{1.9}$$

and α a (yet to be determined) constant.
The path $\mathcal{P}$ between two points x_1^μ and x_2^μ can be parametrised by a parameter τ. With that, the integral of the line element ds becomes an integral over the parameter

$$S = -\alpha \int_{\tau_1}^{\tau_2} d\tau \sqrt{-\eta_{\mu\nu} \frac{\partial x^\mu}{\partial \tau} \frac{\partial x^\nu}{\partial \tau}} \,. \tag{1.10}$$

a) Parametrise the path by the time coordinate $t = x^0$ and take the non-relativistic limit $|\dot{\vec{x}}| \ll 1$ to determine the value of the constant α.

b) Derive the equations of motion by varying the action. *Hint:* You may want to determine the canonically conjugate momentum first.

2.1. Integral definition of the step function

In this exercise we will demonstrate that:

$$\frac{d}{dx}\,\theta(x) = \delta(x), \tag{2.1}$$

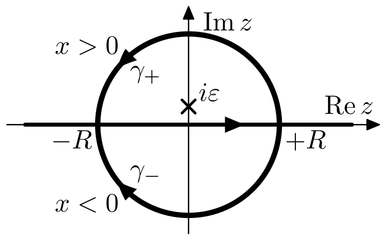
where:

$$\theta(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x < 0, \end{cases} \qquad \int_A dx\,\delta(x)\,f(x) = \begin{cases} f(0) & \text{if } 0 \in A, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

To that end, consider the following function (where $x, z \in \mathbb{R}$, $\varepsilon > 0$):

$$F(x,\varepsilon) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} dz\,\frac{e^{ixz}}{z - i\varepsilon}\,. \tag{2.3}$$

a) Consider a semi-circular path $\gamma_\pm(R)$ of radius R in the upper/lower half of the complex plane and ending on the real axis.



(2.4)

Argue that:

$$\begin{aligned} \lim_{R\to\infty} \int_{\gamma_+(R)} dz\,\frac{e^{ixz}}{z-i\varepsilon} &= 0 \quad \text{if } x > 0, \\ \lim_{R\to\infty} \int_{\gamma_-(R)} dz\,\frac{e^{ixz}}{z-i\varepsilon} &= 0 \quad \text{if } x < 0. \end{aligned} \tag{2.5}$$

Hint: Use integration by parts to improve convergence of the integral.

b) Consider the closed path $\Gamma = [-R,+R] \cup \gamma_\pm(R)$ and make use of the Cauchy integral formula:

$$\frac{1}{2\pi i}\oint_\Gamma dz\,f(z) = \operatorname{Res}_\Gamma f, \tag{2.6}$$

where $\operatorname{Res}_\Gamma f$ is the sum of the residues of the poles of f surrounded by the contour Γ, to show that:

$$\lim_{\varepsilon\to 0^+} F(x,\varepsilon) = \theta(x). \tag{2.7}$$

c) Finally, using (2.3), show relation (2.1). You will have to perform some mathematically questionable steps. Which are they precisely? Can they be justified? How?

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2.2. Discrete and continuous treatment of a 1D spring lattice

Consider a one-dimensional array of N particles at positions $q_i(t)$, $i = 1, \ldots, N$ connected by elastic springs with spring force constant κ. Assume that all of the particles have mass m, and at rest their relative distance is a.

a) Derive the Lagrangian $L(q_i(t), \dot{q}_i(t))$ of this system and compute the Euler–Lagrange equations.

b) Determine the continuum form of these equations by taking the limit $a \to 0$ and $N \to \infty$, where the mass density $\mu := \lim_{a\to 0}(m/a)$ and the elastic modulus $Y := \lim_{a\to 0}(\kappa a)$ are kept fixed, and: $q_i(t) \to \phi(x_0 + ia, t)$ where $i \in \mathbb{Z}$.

c) Directly take the continuum limit of $L(q_i(t), \dot{q}_i(t))$ and show that the Euler–Lagrange equations of the Lagrangian density $\mathcal{L}(\phi, \phi', \dot{\phi})$ are the same as those obtained in part b).

2.3. Classical field momentum

Consider the Lagrangian of a real scalar field $\phi = \phi(x)$:

$$\mathcal{L} = -\tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2. \tag{2.8}$$

a) Write down the stress-energy tensor of the theory using the general result obtained in problem 1.2.

b) Derive:

$$P^\mu = \int \frac{d^3\vec{p}}{(2\pi)^3\,2e(\vec{p})}\,p^\mu(\vec{p})\,a^*(\vec{p})a(\vec{p}) \tag{2.9}$$

by using the definition: $P^\mu = \int d^3\vec{x}\,T^{0\mu}$.

c) Calculate the Poisson bracket $\{P^\mu, \phi(\vec{x})\}$ and interpret the result.

3.1. Scalar field correlator

In this problem we shall consider the amplitude for a particle to be created at point y and annihilated at point x

$$\mathit{\Delta}_+(y,x) := i\langle 0|\phi(y)\phi(x)|0\rangle. \tag{3.1}$$

a) Use the Fourier expansion of $\phi(x)$ to show that

$$\mathit{\Delta}_+(y,x) = i\int \frac{d^3\vec{p}}{(2\pi)^3\, 2e(\vec{p})}\, e^{ip\cdot(y-x)}\,, \tag{3.2}$$

where $p^0 = e(\vec{p}) := \sqrt{\vec{p}^{\,2}+m^2}$.

b) Observe that the amplitude for the propagation of a particle from y to x satisfies

$$\mathit{\Delta}_+(y,x) = \mathit{\Delta}_+(y-x,0) =: \mathit{\Delta}_+(y-x). \tag{3.3}$$

What are the properties of $\mathit{\Delta}_+$ under translations and Lorentz transformations?

c) Express $\mathit{\Delta}_+(x)$ for a time-like x as a single integral over the energy, and the one for space-like x as a single integral over $p = |\vec{p}|$.

Hint: Use a Lorentz transformation to reduce to the cases $\vec{x} = 0$ and $x^0 = 0$ respectively.

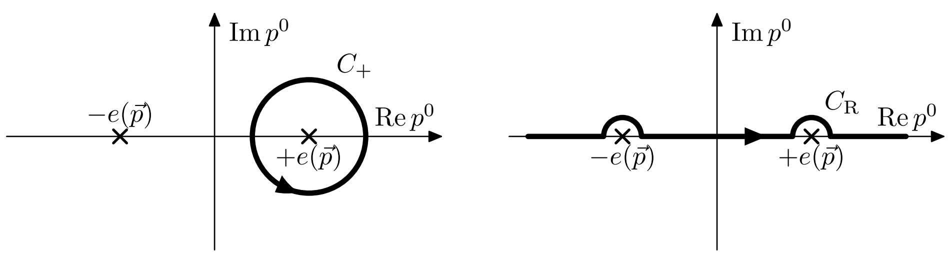
d) Use Cauchy's residue theorem to show that $\mathit{\Delta}_+(x)$ can be also written as

$$\mathit{\Delta}_+(x) = -\int_{C_+} \frac{d^4p}{(2\pi)^4}\,\frac{e^{ip\cdot x}}{p^2+m^2}\,, \tag{3.4}$$

where the integration over the contour C_+ given in the left figure of (3.6) corresponds to the (complex) variable p^0.

e) Show that $\mathit{\Delta}_+(x)$ satisfies the Klein–Gordon equation, i.e.

$$(-\partial^2 + m^2)\mathit{\Delta}_+(x) = 0. \tag{3.5}$$



(3.6)

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3.2. Commutator and Causality

In order to know whether a measurement of the field at x can affect another measurement at y, one may compute the commutator

$$\varDelta(y-x) := i\langle 0|[\phi(y), \phi(x)]|0\rangle = \varDelta_+(y-x) - \varDelta_+(x-y). \tag{3.7}$$

Show that such a commutator vanishes for a space-like separation of x and y, which proves that causality is obeyed.

3.3. Retarded propagator

Now consider the commutator (3.7) and define

$$G_{\mathrm{R}}(y-x) := \theta(y^0 - x^0)\varDelta(y-x), \tag{3.8}$$

which clearly vanishes for any $y^0 < x^0$.

a) Show that

$$G_{\mathrm{R}}(x) = \int_{C_{\mathrm{R}}} \frac{d^4p}{(2\pi)^4} \frac{e^{ip\cdot x}}{p^2+m^2}\ , \tag{3.9}$$

with the contour C_{R} given in the right figure of (3.6).

b) Check that G_{R} is a Green function for the Klein–Gordon equation,

$$(-\partial^2 + m^2)G_{\mathrm{R}}(x) = \delta^4(x). \tag{3.10}$$

3.4. Complex scalar field

We want to investigate the theory of a complex scalar field $\phi = \phi(x)$. The theory is described by the Lagrangian density:

$$\mathcal{L} = -\partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi. \tag{3.11}$$

As a complex scalar field has two degrees of freedom, we can treat ϕ and ϕ^* as independent fields with one degree of freedom each.

a) Find the conjugate momenta $\pi(\vec{x})$ and $\pi^*(\vec{x})$ to $\phi(\vec{x})$ and $\phi^*(\vec{x})$ and the canonical commutation relations. *Note:* we choose $\pi = \partial\mathcal{L}/\partial\dot{\phi}$ rather than $\pi = \partial\mathcal{L}/\partial\dot{\phi}^*$.

b) Find the Hamiltonian of the theory.

c) Introduce creation and annihilation operators to diagonalise the Hamiltonian.

d) Show that the theory contains two sets of particles of mass m.

e) Consider the conserved charge

$$Q = -\frac{i}{2}\int d^3\vec{x}\,(\pi\phi - \phi^*\pi^*). \tag{3.12}$$

Rewrite it in terms of ladder operators and determine the charges of the two particle species.

Quantum Field Theory I — Problem Set 4

ETH Zurich, HS14 — Prof. N. Beisert

4.1. Representations of the Lorentz algebra

The Lie algebra of the Lorentz group is given by

$$[M^{\mu\nu}, M^{\lambda\kappa}] = i(\eta^{\mu\kappa}\, M^{\nu\lambda} + \eta^{\nu\lambda}\, M^{\mu\kappa} - \eta^{\nu\kappa}\, M^{\mu\lambda} - \eta^{\mu\lambda}\, M^{\nu\kappa}). \tag{4.1}$$

a) Show that the generators of the vector representation

$$(J^{\mu\nu})^{\rho}{}_{\sigma} \equiv i(\eta^{\mu\rho}\, \delta^{\nu}_{\sigma} - \eta^{\nu\rho}\, \delta^{\mu}_{\sigma}) \tag{4.2}$$

satisfy the Lie algebra (4.1).

b) Show that the operators

$$L^{\mu\nu} \equiv i(x^{\mu}\, \partial^{\nu} - x^{\nu}\, \partial^{\mu}) \tag{4.3}$$

satisfy the Lie algebra (4.1).

4.2. Conservation of charge with complex scalar fields

Consider a free complex scalar field described by

$$\mathcal{L} = -(\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi. \tag{4.4}$$

a) Show that the transformation

$$\phi(x) \longrightarrow \phi'(x) = e^{i\alpha}\phi(x) \tag{4.5}$$

leaves the Lagrangian density invariant.

b) Find the conserved current associated with this symmetry.

If we now consider two complex scalar fields, the Lagrangian density is given by

$$\mathcal{L} = -(\partial_\mu \phi^*_a)(\partial^\mu \phi^a) - m^2 \phi^*_a \phi^a, \qquad a = 1, 2. \tag{4.6}$$

c) Show that

$$\phi^a(x) \longrightarrow \phi'^{\,a}(x) = U^a{}_b \phi^b(x) \tag{4.7}$$

with $U \in \mathrm{U}(2) = \left\{A \in \mathbb{C}^{2\times 2} : A^{-1} = A^\dagger = (A^*)^\mathsf{T}\right\}$ is a symmetry transformation.

d) Show that now there are four conserved charges: one given by the generalisation of part b), and the other three given by

$$Q_i = \frac{i}{2}\int d^3\vec{x}\,\left(\phi^*_a (\sigma^i)^a{}_b \pi^{*\,b} - \pi_a (\sigma^i)^a{}_b \phi^b\right), \tag{4.8}$$

where σ^i are the Pauli matrices.

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4.3. Symmetry of the stress-energy tensor

Consider a relativistic scalar field theory specified by some Lagrangian $\mathcal{L}(\phi, \partial\phi)$.

a) Compute the variation of $\mathcal{L}(\phi(x), \partial\phi(x))$ under infinitesimal Lorentz transformations (note: $\omega^{\mu\nu} = -\omega^{\nu\mu}$)

$$x^\mu \longrightarrow x^\mu - \omega^\mu{}_\nu x^\nu. \tag{4.9}$$

b) Assuming that $\mathcal{L}(x)$ transforms as a scalar field, i.e. just like $\phi(x)$, derive another expression for its variation under Lorentz transformations.

c) Compare the two expressions to show that the two indices of the stress-energy tensor are symmetric

$$T^{\mu\nu} = -\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\,\partial^\nu\phi + \eta^{\mu\nu}\mathcal{L} = T^{\nu\mu}. \tag{4.10}$$

5.1. Properties of gamma-matrices

The gamma-matrices satisfy a Clifford algebra,

$$\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}\,\mathrm{id}\,. \tag{5.1}$$

Note: The minus sign in the Clifford algebra is a matter of convention.

a) Show the following contraction identities using (5.1):

$$\begin{aligned} \gamma^\mu\gamma_\mu &= -4\,\mathrm{id}, & (5.2)\\ \gamma^\mu\gamma^\nu\gamma_\mu &= 2\gamma^\nu, & (5.3)\\ \gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu &= 4\eta^{\nu\rho}\,\mathrm{id}, & (5.4)\\ \gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu &= 2\gamma^\sigma\gamma^\rho\gamma^\nu. & (5.5) \end{aligned}$$

b) Show the following trace properties using (5.1):

$$\begin{aligned} \mathrm{tr}(\gamma^{\mu_1}\cdots\gamma^{\mu_n}) &= 0 \text{ for odd } n, & (5.6)\\ \mathrm{tr}(\gamma^\mu\gamma^\nu) &= -4\eta^{\mu\nu}, & (5.7)\\ \mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}). & (5.8) \end{aligned}$$

5.2. Dirac and Weyl representations of the gamma-matrices

Using the Pauli matrices σ^i, $i = 1, 2, 3$, together with the 2×2 identity matrix σ^0,

$$\sigma^0 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{5.9}$$

we can realise the Dirac representation of the gamma-matrices,

$$\gamma_{\mathrm{D}}^0 \equiv \sigma^0 \otimes \sigma^3, \qquad \gamma_{\mathrm{D}}^j \equiv \sigma^j \otimes i\sigma^2 \quad (j = 1, 2, 3), \tag{5.10}$$

where the tensor product can be written as a 4×4 matrix in 2×2 block form as follows

$$A \otimes B \equiv \begin{pmatrix} B_{11}A & B_{12}A \\ B_{21}A & B_{22}A \end{pmatrix}. \tag{5.11}$$

Denoting the Pauli matrices collectively by σ^μ and defining $(\bar\sigma^0, \bar\sigma^i) = (\sigma^0, -\sigma^i)$. We can then define the gamma-matrices in the Weyl representation:

$$\gamma_{\mathrm{W}}^\mu \equiv \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}. \tag{5.12}$$

Show that both representations satisfy the Clifford algebra (5.1). Can you show their equivalence, i.e. $\gamma_{\mathrm{W}}^\mu = T\gamma_{\mathrm{D}}^\mu T^{-1}$ for some matrix T?

Hint: It may help to write the γ_{W}^μ as tensor products of Pauli matrices.

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5.3. Spinor rotations

The Dirac equation is invariant under Lorentz transformations $\Psi'(x') = S\Psi(x)$ if the spinor transformation matrix S satisfies

$$\Lambda^{\mu}{}_{\nu}\, S^{-1}\gamma^{\nu} S = \gamma^{\mu}. \tag{5.13}$$

For an infinitesimal Lorentz transformation $\Lambda_{\mu\nu} = \eta_{\mu\nu} + \delta\omega_{\mu\nu}$ this is fulfilled if

$$S = 1 + \tfrac{1}{8}\delta\omega_{\mu\nu}[\gamma^{\mu}, \gamma^{\nu}]. \tag{5.14}$$

a) Find the infinitesimal spinor transformation δS for a rotation around the z-axis, i.e. only non-zero components of $\delta\omega_{\mu\nu}$ are $\delta\omega_{12} = -\delta\omega_{21} \neq 0$.

b) Finite transformations are obtained by considering a consecutive application of infinitely many, $N \to \infty$, infinitesimal transformations $\delta\omega = \omega/N$

$$S = \lim_{N\to\infty} \left(1 + \frac{1}{8}\frac{\omega_{\mu\nu}}{N}[\gamma^{\mu}, \gamma^{\nu}]\right)^{N} = \exp\bigl(\tfrac{1}{8}\omega_{\mu\nu}[\gamma^{\mu}, \gamma^{\nu}]\bigr). \tag{5.15}$$

Compute the finite rotation with angle ω_{12} around the same axis as before. Also compute the finite transformation $\Lambda = \exp(\omega)$ for vectors.

c) What happens to the individual components of a spinor under this transformation? What is the period of the transformation in the angle ω_{12}? Compare it to the finite rotation for vectors.

6.1. Completeness for gamma-matrices

An arbitrary product of gamma-matrices is proportional to one of the following 16 linearly independent matrices $\mathit{\Gamma}^a$ (here a is a multi-index which specifies the type of matrix, $\mathrm{S}, \mathrm{P}, \mathrm{V}, \mathrm{A}, \mathrm{T}$, along the corresponding indices if any)

$$\mathit{\Gamma}^{\mathrm{S}} = 1, \tag{6.1}$$

$$\mathit{\Gamma}^{\mathrm{P}} = \gamma^5, \tag{6.2}$$

$$\mathit{\Gamma}^{\mathrm{V},\mu} = \gamma^\mu, \tag{6.3}$$

$$\mathit{\Gamma}^{\mathrm{A},\mu} = i\gamma^5\gamma^\mu, \tag{6.4}$$

$$\mathit{\Gamma}^{\mathrm{T},\mu\nu} = \tfrac{i}{2}[\gamma^\mu, \gamma^\nu]. \tag{6.5}$$

a) Show that the trace of any product of $\mathit{\Gamma}$'s is given by $\operatorname{tr}(\mathit{\Gamma}^a\mathit{\Gamma}^b) = \pm 4\delta^{ab}$. For simplicity we ignore the signs arising from the Lorentz signature.

b) Show that for any $a \neq b$ there is a $n \neq \mathrm{S}$ such that $\mathit{\Gamma}^a\mathit{\Gamma}^b = \alpha\mathit{\Gamma}^n$ with some $\alpha \in \mathbb{C}$.

c) Show that the matrices are linearly independent and therefor form a complete basis of 4×4 spinor matrices. *Hint:* To do this consider a sum $\sum_a \alpha_a \mathit{\Gamma}^a = 0$. What can be said about the coefficients?

6.2. Spinors, spin sums and completeness relations

In this exercise we will use the Weyl representation (5.12) defined in the previous exercise sheet.

a) Show that $(p \cdot \sigma)(p \cdot \bar\sigma) = -p^2$.

b) Prove that the below 4-spinor $u_\alpha(\vec p)$ solves Dirac's equation $(p_\mu\gamma^\mu + m\,\mathrm{id})u_\alpha(\vec p) = 0$

$$u_\alpha(\vec p) = \begin{pmatrix} \sqrt{-p\cdot\sigma}\,\xi_\alpha \\ \sqrt{-p\cdot\bar\sigma}\,\xi_\alpha \end{pmatrix}, \tag{6.6}$$

where $\xi_\pm$ form a basis of 2-spinors.

c) Suppose, the 2-spinors ξ_+ and ξ_- are orthonormal. What does it imply for $\xi_\alpha^\dagger\xi_\alpha$ and

$$\sum_{\alpha\in\{+,-\}} \xi_\alpha\xi_\alpha^\dagger\,? \tag{6.7}$$

d) Show that $\bar u_\alpha(\vec p)u_\alpha(\vec p) = 2m$ for $\alpha \in \{+,-\}$.

e) Show the completeness relation:

$$\sum_{\alpha\in\{+,-\}} u_\alpha(\vec p)\bar u_\alpha(\vec p) = -p_\mu\gamma^\mu + m\,\mathrm{id}\,. \tag{6.8}$$

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6.3. Gordon identity

Prove the Gordon identity,

$$\bar{u}_\beta(\vec{q})\gamma^\mu u_\alpha(\vec{p}) = \frac{1}{2m}\,\bar{u}_\beta(\vec{q})\big[\mathrm{id}(q+p)^\mu + \tfrac{1}{2}[\gamma^\mu,\gamma^\nu](q-p)_\nu\big]u_\alpha(\vec{p}). \tag{6.9}$$

Hint: You can do this using just $\{\gamma^\mu,\gamma^\nu\} = -2\eta^{\mu\nu}\,\mathrm{id}$.

6.4. Fierz identity

a) Use the linear independence of the Γ^a matrices to show that

$$\delta^\alpha_\gamma\delta^\beta_\delta = \sum_i \frac{1}{4}(\Gamma_i)^\alpha{}_\delta(\Gamma_i)^\beta{}_\gamma. \tag{6.10}$$

Hint: Decompose an arbitrary matrix $M^\alpha{}_\beta = \sum_i m_i(\Gamma^i)^\alpha{}_\beta$ and find the coefficients m_i.

b) Use the result from a) to show the Fierz identity:

$$(\Gamma^i)^\alpha{}_\beta(\Gamma^j)^\gamma{}_\delta = \sum_{k,l}\frac{1}{16}\,\mathrm{tr}(\Gamma^i\Gamma^l\Gamma^j\Gamma^k)(\Gamma^k)^\alpha{}_\delta(\Gamma^l)^\gamma{}_\beta. \tag{6.11}$$

c) Find the Fierz transformation for the spinor products

$$(\bar{u}_1u_2)(\bar{u}_3u_4) \qquad \text{and} \qquad (\bar{u}_1\gamma^\mu u_2)(\bar{u}_3\gamma_\mu u_4). \tag{6.12}$$

7.1. Helicity and chirality

In four dimensions we can define the chirality operator as

$$\gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3. \tag{7.1}$$

a) Show that γ^5 satisfies

$$\{\gamma^5, \gamma^\mu\} = 0, \qquad \left(\gamma^5\right)^2 = 1. \tag{7.2}$$

b) Show that the operators

$$P_{\mathrm{R,L}} = \frac{1 \pm \gamma_5}{2}, \tag{7.3}$$

are two orthogonal projectors to the chiral subspaces and that they satisfy the completeness relation

$$P_{\mathrm{L}} + P_{\mathrm{R}} = 1. \tag{7.4}$$

Helicity is defined to be the projection of spin along the direction of motion,

$$h(\vec{p}) = \frac{\vec{\Sigma} \cdot \vec{p}}{|\vec{p}|}. \tag{7.5}$$

Here, $\vec{\Sigma}$ is the spin operator which is given in the Weyl representation by

$$\vec{\Sigma} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}. \tag{7.6}$$

c) Show that helicity and chirality are equivalent for a massless spinor $u_s(\vec{p})$.

Now consider the Dirac Lagrangian

$$\mathcal{L} = \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi. \tag{7.7}$$

d) Find the corresponding Hamiltonian.

e) Show that chirality is not conserved for a massive fermion by computing the equations of motions for the chiral fermions ψ_{L} and ψ_{R}, with

$$\psi_{\mathrm{L,R}} = P_{\mathrm{L,R}}\psi. \tag{7.8}$$

f) Show that helicity is conserved, then argue it is not Lorentz invariant for $m \neq 0$.

g) Show that for $m = 0$ the Dirac Lagrangian is invariant under a chiral transformation $U = \exp(-i\alpha\gamma^5)$ of the fields, and derive the associated conserved current. Show that having a non-zero mass breaks the symmetry.

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7.2. Electrodynamics

Consider the Lagrange density

$$\mathcal{L}(A_\mu) = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - J^\mu A_\mu\,, \qquad \text{where} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,, \tag{7.9}$$

and J^μ is some external source field.

a) Show that the Euler–Lagrange equations are the inhomogeneous Maxwell equations. The usual electromagnetic fields are defined by $E^i = -F^{0i}$ and $\varepsilon^{ijk}B^k = -F^{ij}$. What about the homogeneous Maxwell equations?

b) Construct the stress-energy tensor for this theory.

c) Convince yourself that the stress-energy tensor is not symmetric. In order to make it symmetric consider

$$\hat{T}^{\mu\nu} = T^{\mu\nu} - \partial_\lambda K^{\lambda\mu,\nu}\,, \tag{7.10}$$

where $K^{\lambda\mu,\nu}$ is anti-symmetric in the first two indices. By taking

$$K^{\lambda\mu,\nu} = F^{\mu\lambda}A^\nu \tag{7.11}$$

show that the modified stress-energy tensor $\hat{T}^{\mu\nu}$ is symmetric, and that it leads to the standard formulae for the electromagnetic energy and momentum densities

$$\mathcal{E} = \tfrac{1}{2}(\vec{E}^{\,2} + \vec{B}^{\,2})\,, \qquad \vec{\mathcal{S}} = \vec{E}\times\vec{B}\,. \tag{7.12}$$

d) *for fun:* Show that all Maxwell equations can summarised as

$$\gamma^\nu\gamma^\rho\gamma^\sigma\partial_\nu F_{\rho\sigma} = -2\gamma^\nu J_\nu. \tag{7.13}$$

8.1. The massive vector field

Consider the Lagrangian for the free massive spin-1 field V_μ:

$$\mathcal{L} = -\tfrac{1}{2}\partial^\mu V^\nu \partial_\mu V_\nu + \tfrac{1}{2}\partial^\mu V^\nu \partial_\nu V_\mu - \tfrac{1}{2}m^2 V^\mu V_\mu. \tag{8.1}$$

a) Derive the Euler–Lagrange equations of motion for V_μ.

b) By taking a derivative of the equation, show that V_μ is a conserved current.

c) Show that V_μ satisfies the Klein–Gordon equation.

8.2. Hamiltonian formulation

The Hamiltonian formulation of the massive vector is somewhat tedious due to the presence of constraints.

a) Derive the momenta Π_μ conjugate to the fields V_μ. Considering the space and time components separately, what do you notice?

Your observation is related to constraints. The time component V_0 of the vector field is completely determined by the spatial components and their conjugate momenta (without making reference to time derivatives).

b) Use the equations derived in problem 8.1 to show that

$$V_0 = -m^{-2}\partial_k \Pi_k, \qquad \dot{V}_0 = \partial_k V_k. \tag{8.2}$$

c) Substitute this solution for V_0 and $\dot{V}_0$ into the Lagrangian and perform a Legendre transformation to obtain the Hamiltonian. Show that

$$H = \int d^3\vec{x}\,\Big(\tfrac{1}{2}\Pi_k\Pi_k + \tfrac{1}{2}m^{-2}\partial_k\Pi_k\partial_l\Pi_l + \tfrac{1}{2}\partial_k V_l \partial_k V_l - \tfrac{1}{2}\partial_l V_k \partial_k V_l + \tfrac{1}{2}m^2 V_k V_k\Big). \tag{8.3}$$

d) Derive the Hamiltonian equations of motion for V_k and Π_k, and compare them to the results of problem 8.1.

8.3. Commutators

The unequal time commutators $[V_\mu(x), V_\nu(y)] = -i\Delta^{\mathrm{V}}_{\mu\nu}(x-y)$ for the massive vector field read

$$\Delta^{\mathrm{V}}_{\mu\nu}(x) = \big(\eta_{\mu\nu} - m^{-2}\partial_\mu\partial_\nu\big)\,\Delta(x), \tag{8.4}$$

where $\Delta(x)$ is the corresponding function for the scalar field.

a) Show that these obey the equations derived in problem 8.1.

b) Show explicitly that they obey the constraint equations in 8.2b), i.e.

$$\big[m^2 V_0(x) + \partial_k\Pi_k(x), V_\nu(y)\big] = \big[\dot{V}_0(x) - \partial_k V_k(x), V_\nu(y)\big] = 0. \tag{8.5}$$

c) Confirm that the equal time commutators take the canonical form

$$[V_k(\vec{x}), V_l(\vec{y})] = [\Pi_k(\vec{x}), \Pi_l(\vec{y})] = 0, \qquad [V_k(\vec{x}), \Pi_l(\vec{y})] = i\delta_{kl}\delta^3(\vec{x}-\vec{y}). \tag{8.6}$$

$\longrightarrow$

8.4. Polarisation vectors of a massless vector field

Each Fourier mode in the plane wave expansion of a massless vector field has the form

$$A_\mu^{(\lambda)}(\vec{p};x) = N(\vec{p})\,\epsilon_\mu^{(\lambda)}(\vec{p})\,e^{ip\cdot x}. \tag{8.7}$$

Without any loss of generality the polarisation vectors $\epsilon_\mu^{(\lambda)}(\vec{p})$ can be chosen to form a four-dimensional orthonormal system satisfying

$$\epsilon_\mu^{(\lambda)}(\vec{p})\,\epsilon^{(\kappa)\mu}(\vec{p}) = \eta^{\lambda\kappa}. \tag{8.8}$$

a) Show that the following choice satisfies (8.8)

$$\epsilon_\mu^{(0)}(\vec{p}) = n_\mu, \tag{8.9}$$
$$\epsilon_\mu^{(1)}(\vec{p}) = (0, \vec{\epsilon}^{\,(1)}(\vec{p})), \tag{8.10}$$
$$\epsilon_\mu^{(2)}(\vec{p}) = (0, \vec{\epsilon}^{\,(2)}(\vec{p})), \tag{8.11}$$
$$\epsilon_\mu^{(3)}(\vec{p}) = (p_\mu + n_\mu(p\cdot n))\,/|p\cdot n|, \tag{8.12}$$

where $n_\mu = (1, \vec{0})$ and $\vec{p}\cdot\vec{\epsilon}^{\,(k)}(\vec{p}) = 0$ as well as $\vec{\epsilon}^{\,(k)}(\vec{p})\cdot\vec{\epsilon}^{\,(l)}(\vec{p}) = \delta^{kl}$.

b) Use the polarisation vectors to verify the completeness relation

$$\sum_{\lambda=0}^{3} \eta_{\lambda\lambda}\,\epsilon_\mu^{(\lambda)}(\vec{p})\,\epsilon_\nu^{(\lambda)}(\vec{p}) = \eta_{\mu\nu}. \tag{8.13}$$

c) Show for the physical modes of the photon that

$$\sum_{\lambda=1}^{2} \epsilon_\mu^{(\lambda)}(\vec{p})\,\epsilon_\nu^{(\lambda)}(\vec{p}) = \eta_{\mu\nu} - \frac{p_\mu p_\nu}{(p\cdot n)^2} - \frac{p_\mu n_\nu + p_\nu n_\mu}{p\cdot n}\,. \tag{8.14}$$

9.1. Photon propagator

The Lagrangian density in the covariant gauge reads

$$\mathcal{L} = -\tfrac{1}{4}\, F_{\mu\nu}(x)\, F^{\mu\nu}(x) - \tfrac{1}{2}\xi\, \bigl(\partial_\mu A^\mu(x)\bigr)^2. \tag{9.1}$$

Show that the photon propagator in the covariant gauge with arbitrary gauge parameter ξ is given by

$$G^{\mathrm{V}}_{\mu\nu}(x-y) = \int \frac{d^4p}{(2\pi)^4}\, \frac{e^{ip(x-y)}}{p^2 - i\epsilon} \left(\eta_{\mu\nu} - (1-\xi^{-1})\frac{p_\mu p_\nu}{p^2}\right). \tag{9.2}$$

Hint: recall that the propagator is the Green function of the equations of motion.

9.2. Evolution operator

The interaction picture field operator $\phi_0(x)$ is related to the full Heisenberg field operator $\phi(x)$ by

$$\phi(t,\vec{x}) = U(t,t_0)^{-1}\phi_0(t,\vec{x})U(t,t_0) \tag{9.3}$$

with

$$U(t,t_0) = \exp\bigl(i(t-t_0)H_0\bigr)\exp\bigl(-i(t-t_0)H\bigr). \tag{9.4}$$

a) Show that $U(t,t_0)$ satisfies the differential equation

$$i\,\frac{\partial}{\partial t} U(t,t_0) = H_{\mathrm{int}}(t)U(t,t_0) \tag{9.5}$$

with the initial condition $U(t_0,t_0) = 1$. Determine the interaction Hamiltonian $H_{\mathrm{int}}(t)$.

b) Show that the unique solution to this equation with the same initial condition can be written as

$$U(t,t_0) = \mathrm{T}\exp\left(-i\int_{t_0}^{t} dt'\, H_{\mathrm{int}}(t')\right). \tag{9.6}$$

c) Show that the operator satisfies $U^\dagger(t,t_0) = U(t_0,t)$ and $U(t_1,t_0)U(t_0,t_2) = U(t_1,t_2)$.

9.3. Wick's theorem

Wick's theorem relates the time-ordered product of fields $\phi_0(x)$ to the normal-ordered product plus all possible contractions

$$\mathrm{T}\bigl[\phi_0(x_1)\ldots\phi_0(x_m)\bigr] = \mathrm{N}\bigl[\phi_0(x_1)\ldots\phi_0(x_m) + \text{all contractions}\bigr]. \tag{9.7}$$

Prove this theorem by induction. What changes in the case of fermionic operators?

10.1. Four-point interaction in scalar QED

Consider a U(1) gauge theory with two complex massive scalar fields ϕ, χ and one vector field A_μ. Each of the scalar fields is coupled to the gauge field and they both have the same charge q. The Lagrangian density of the theory is given by

$$\mathcal{L} = -\tfrac{1}{2}(D_\mu\phi)^\dagger D^\mu\phi - \tfrac{1}{2}(D_\mu\chi)^\dagger D^\mu\chi - \tfrac{1}{2}m^2\phi^\dagger\phi - \tfrac{1}{2}m^2\chi^\dagger\chi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{10.1}$$

with the covariant derivative

$$D_\mu = \partial_\mu - iqA_\mu(x) \tag{10.2}$$

and the associated electromagnetic field strength tensor $F^{\mu\nu}$. Use the Feynman gauge fixing term.

In this exercise we want to compute the first interaction term of ϕ with χ in the expansion in the perturbative parameter q. For this reason we are interested in obtaining the time-ordered four-point correlation function for two fields of type ϕ and two fields of type χ, which is given by

$$\begin{aligned}&\langle 0|\,\mathrm{T}\left\{\phi(x_1)\phi^\dagger(x_2)\chi(x_3)\chi^\dagger(x_4)\right\}|0\rangle_{\text{int}} \\ &\quad= \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|\,\mathrm{T}\left\{\phi(x_1)\phi^\dagger(x_2)\chi(x_3)\chi^\dagger(x_4)\exp\left[i\int_{-T}^{T} d^4x\,\mathcal{L}_{\text{int}}(x)\right]\right\}|0\rangle}{\langle 0|\,\mathrm{T}\left\{\exp\left[i\int_{-T}^{T} d^4x\,\mathcal{L}_{\text{int}}(x)\right]\right\}|0\rangle}.\end{aligned} \tag{10.3}$$

a) Find the interaction Lagrangian $\mathcal{L}_{\text{int}}$ of the theory.

b) What is the leading order of the expansion in q of the time-ordered four-point correlation function that allows for an interaction with the gauge field A_μ?

c) Expand the denominator in q to the order you just found, then use Wick's theorem to decompose it into terms where all fields are contracted. Can you find a diagrammatic representation for the different contributions?

d) Now expand the numerator of the time-ordered four-point correlation function in the same way. It may be useful to draw pictures to simplify the bookkeeping. How can you interpret the terms that do not lead to an interaction of ϕ and χ?

e) Focus on the contribution(s) where ϕ and χ interact non-trivially: Insert the Fourier transformed propagators of the scalar and vector fields into your result

$$\begin{aligned} G_{\mathrm{F}}(x-y) &= i\langle 0|\,\mathrm{T}\{\phi^\dagger(x)\phi(y)\}|0\rangle = \int \frac{d^4p}{(2\pi)^4}\,\frac{e^{ip(x-y)}}{p^2+m^2-i\epsilon}, \\ G_{\mathrm{F}}^{\mu\nu}(x-y) &= i\langle 0|\,\mathrm{T}\{A^\mu(x)A^\nu(y)\}|0\rangle = \int \frac{d^4p}{(2\pi)^4}\,\frac{\eta^{\mu\nu}e^{ip(x-y)}}{p^2-i\epsilon}. \end{aligned} \tag{10.4}$$

f) How do you interpret the limit of $T\to\infty$ in equation (10.3)? Simplify your result by performing the integration over the internal variables. How can you interpret the individual factors in your result?

g) *Optional:* How will your result in f) change if you use a different gauge? E.g. use a Lorenz gauge fixing term with $\xi \neq 1$. *Hint:* The gauge affects only $G_{\mathrm{F}}^{\mu\nu}$.

11.1. Optical theorem

Write the scattering matrix S in terms of its matrix elements M as

$$S_{ij} = \delta_{ij} + (2\pi)^4 \delta^4(p_i - p_j) i M_{ij} \tag{11.1}$$

with the indices i, j enumerating a basis of Fock space (we shall glance over the fact that Fock space is infinite-dimensional).

Use unitarity of the scattering matrix, $S^\dagger S = 1$, to show the unitarity relation for matrix elements

$$M_{fi} - M^*_{if} = i \sum_n (2\pi)^4 \delta^4(p_f - p_n) M_{fn} M^*_{in}. \tag{11.2}$$

11.2. Møller scattering

a) Calculate the $\mathcal{O}(q^2)$ contribution to the scattering matrix element for Møller scattering:

$$e^-(p_1, \alpha_1) + e^-(p_2, \alpha_2) \longrightarrow e^-(p_3, \alpha_3) + e^-(p_4, \alpha_4) \tag{11.3}$$

through direct evaluation in position space.

b) Repeat the calculation in part a) using the Feynman rules for QED in momentum space.

11.3. Kinematics in 2 → 2 scattering

Consider a $2 \to 2$ particle scattering process with the kinematics $p_1 + p_2 \to p_3 + p_4$.

a) Show that in the centre-of-mass frame the energies $e(\vec{p}_i)$ and the norms of momenta $|\vec{p}_i|$ of the incoming and the outgoing particles are entirely fixed by the total centre-of-mass energy s and the particle masses m_i.

b) Show that the scattering angle θ between $\vec{p}_1$ and $\vec{p}_3$ is given by

$$\theta = \arccos\left(\frac{s(t-u) + (m_1^2 - m_2^2)(m_3^2 - m_4^2)}{\sqrt{\lambda(s, m_1^2, m_2^2)}\sqrt{\lambda(s, m_3^2, m_4^2)}}\right), \tag{11.4}$$

with the Mandelstam variables given by

$$s = -(p_1 + p_2)^2, \qquad t = -(p_1 - p_3)^2, \qquad u = -(p_1 - p_4)^2, \tag{11.5}$$

and the Källén function defined as

$$\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz. \tag{11.6}$$

c) Show that $s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$.

d) Determine $t_{\min}$ and $t_{\max}$ from the condition $|\cos\theta| \le 1$, and study the behaviour of $t_{\min}$ and $t_{\max}$ in the limit $s \gg m_i^2$.

12.1. Volume of higher-dimensional spheres

The integrands of D-dimensional integrals often are functions $F(\vec{x}) = F(|\vec{x}|)$ with spherical symmetry (or they can be brought into this form). The angular part of the integral in spherical coordinates yields the volume of the $(D-1)$-dimensional sphere S^{D-1}

$$\int d^D\vec{x}\, F(|\vec{x}|) = \mathrm{Vol}(S^{D-1}) \int_0^\infty r^{D-1} dr\, F(r). \tag{12.1}$$

In particular, in view of the dimensional regularisation scheme, where D is assumed to be a real number, we need a suitable formula for the volume as an analytic function of D. Use the well-known result

$$\int_{-\infty}^{\infty} dx\, \exp(-x^2) = \sqrt{\pi}, \tag{12.2}$$

to show that the volume of the $(D-1)$-sphere is

$$\mathrm{Vol}(S^{D-1}) = \frac{2\pi^{D/2}}{\Gamma(D/2)}\,. \tag{12.3}$$

12.2. Muon pair production

Follow the steps below to calculate the total cross section for the process $e^+e^- \to \mu^+\mu^-$.

a) Draw all the diagrams that contribute to this process at the lowest non-trivial order, and use the Feynman rules for QED in momentum space to obtain the scattering amplitude M.

b) Compute $|M|^2$. Assuming that the particle spins are not measured, sum over the spins of the outgoing particle, and average over those of the incoming ones. This should help you bring your expression for $|M|^2$ into a much simpler form. *Hint:* You might find the completeness relations for spinors useful.

c) The differential cross section in the centre-of-mass frame is given by

$$d\sigma = \frac{|M|^2}{4|\vec{p}_1|\sqrt{s}} \frac{d^3\vec{p}_3}{(2\pi)^3\, 2e(\vec{p}_3)} \frac{d^3\vec{p}_4}{(2\pi)^3\, 2e(\vec{p}_4)} (2\pi)^4\delta^4(p_1+p_2-p_3-p_4). \tag{12.4}$$

Use the result for $|M|^2$ that you obtained above, and integrate over $\vec{p}_3$ and $\vec{p}_4$ to obtain the total cross section $\sigma = \int d\sigma$.

13.1. Feynman and Schwinger parameters

a) To evaluate loop diagrams one combines propagators with the use of *Feynman parameters*. The basic version is

$$\frac{1}{AB} = \int_0^1 \frac{dx}{[xA + (1-x)B]^2}, \tag{13.1}$$

but it can be generalised to n propagators elevated to some arbitrary power

$$\frac{1}{\prod_{i=1}^n A_i^{\nu_i}} = \frac{\Gamma\left(\sum_{i=1}^n \nu_i\right)}{\prod_{i=1}^n \Gamma(\nu_i)} \int_0^1 \left(\prod_{i=1}^n dx_i\right) \delta\left(1 - \sum_{i=1}^n x_i\right) \frac{\prod_{i=1}^n x_i^{\nu_i - 1}}{\left[\sum_{i=1}^n x_i A_i\right]^{\sum_{i=1}^n \nu_i}}. \tag{13.2}$$

Prove (13.2) recursively.

b) Another useful parametrisation is the *Schwinger parametrisation*:

$$\frac{1}{A^\nu} = \frac{1}{\Gamma(\nu)} \int_0^\infty d\alpha\, \alpha^{\nu-1} e^{-\alpha A}. \tag{13.3}$$

Prove (13.3).

13.2. Electron self energy structure

In QED, the electron two-point function $F(p,q) = -i(2\pi)^4 \delta^4(p+q) M(p)$ receives contributions from self energy diagrams.

a) Draw the Feynman diagrams corresponding to the one- and two-loop contributions. Which of these diagrams are one-particle irreducible?

b) For the one-loop case, write down the expression for $M(p)$ using the massive QED Feynman rules in momentum space and argue why the integral is divergent.

c) Explain why one can make the ansatz

$$M = p \cdot \gamma\, M_{\mathrm{V}} + m\, M_{\mathrm{S}}, \tag{13.4}$$

where $M_{\mathrm{V,S}}$ are scalar functions. Write down integral expressions for them.

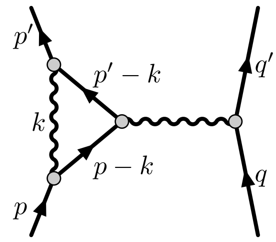
14.1. A one-loop correction to scattering in QED

The aim of this exercise is to gain an insight into the calculation of loop corrections to scattering amplitudes. To this end consider the one-loop corrections to $e^-e^- \to e^-e^-$ scattering in QED.

a) Draw all amputated and connected graphs that would contribute to this process. You should find ten different contributions.

b) How does the field strength renormalisation factor for the spinors, $Z_\psi = 1 + Z_\psi^{(2)} + \ldots$, contribute at this perturbative order? How does the field strength renormalisation of the photon Z_A contribute to the process? Can you sketch suitable Feynman graphs?

Now focus on the following diagram:



(14.1)

c) Write the scattering matrix element corresponding to the amputated Feynman graph, and bring it to the following form

$$iM = q^4\, \bar{v}(\vec{q})\gamma_\mu v(\vec{q}\,')\, \frac{1}{(p-p')^2 - i\epsilon} \int \frac{d^D k}{(2\pi)^D}\, \bar{v}(\vec{p})\, \frac{A^\mu}{B}\, v(\vec{p}\,'). \tag{14.2}$$

d) Use a suitable Feynman parametrisation to rewrite the denominator B as

$$\frac{1}{B} = 2\int_0^1 dx \int_0^1 dy \int_0^1 dz\, \frac{\delta(1-x-y-z)}{C^3}\,, \tag{14.3}$$

where

$$C = k^2 - 2k\cdot(xp + yp') - i\epsilon. \tag{14.4}$$

Complete the square and show that C can be written as

$$C = k'^2 + (1-z)^2 m^2 + xy\,(p-p')^2 - i\epsilon, \qquad k' = k - xp - yp'. \tag{14.5}$$

$\longrightarrow$

e) Show that the numerator A^μ can be brought to the form

$$A^\mu = \left[k'^2 + 2(1-4z+z^2)m^2 - 2(z+xy)(p-p')^2\right]\gamma^\mu \\ + z(1-z)m[\gamma^\mu,\gamma^\nu](p'-p)_\nu. \tag{14.6}$$

To do so, use:

- the anti-commutation relations for gamma-matrices

$$(p\cdot\gamma)\gamma^\mu = -2p^\mu - \gamma^\mu(\gamma\cdot p), \tag{14.7}$$

- the Dirac equation,

$$\bar{v}(\vec{p})(\gamma\cdot p) = m\bar{v}(\vec{p}), \qquad (p'\cdot\gamma)v(\vec{p}\,') = mv(\vec{p}\,'), \tag{14.8}$$

- the symmetry of the integration over k', which allows the following tensorial replacements in the numerator

$$k'^\mu \to 0, \qquad k'^\mu k'^\nu \to \frac{1}{D}\,\eta^{\mu\nu}k'^2, \tag{14.9}$$

- the symmetry of the integral under the interchange $x \leftrightarrow y$,
- the Gordon identity

$$\bar{v}(\vec{p})\gamma^\mu v(\vec{p}\,') = \frac{1}{2m}\,\bar{v}(\vec{p})\left[-(p+p')^\mu - \tfrac{1}{2}[\gamma^\mu,\gamma^\nu](p-p')_\nu\right]v(\vec{p}\,'). \tag{14.10}$$

For the remainder of this problem, you may assume that the virtuality of the photon is small, $|(p-p')^2| \ll m^2$.

f) Using the results obtained in problem 13.2, integrate over the loop momentum k'. *Note:* Split off a divergent contribution, and cut off the integral as discussed in the lecture. Can you interpret the residual dependence on the cutoff?

g) Integrate over x and y. *Note:* Cut off the integral if needed. Can you interpret the residual dependence on the cutoff?