Quantum Information Theory
Series 7

FS 14
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Exercise 7.1 Entropy properties

The von-Neumann entropy is defined as:

$$H(\rho) = -tr[\rho \log \rho] = -\sum_k \lambda_k \log \lambda_k$$

Show the following properties of this quantity:

1. $H(\rho) = 0$ iff ρ is pure
2. $H(U\rho U^*) = H(\rho)$ for unitary U
3. $H(\rho) \leq \log |supp\, \rho|$
4. $H(\sum_k p_k \rho_k) \geq \sum_k p_k H(\rho_k)$
5. $H(\sum_k P_k \rho P_k) \geq H(\rho)$ for any complete set of projectors P_k

Exercise 7.2 Geometry of Measurements

In this exercise we will learn that the set of 2-outcome POVMs is a convex set with orthogonal measurements as extremal points.
Let $F = \{F_1, F_2\}$ and $G = \{G_1, G_2\}$ be two POVMs. We define an element-wise convex combination of F and G as $\alpha F + (1-\alpha)G := \{\alpha F_1 + (1-\alpha)G_1, \alpha F_2 + (1-\alpha)G_2\}$, with $0 \leq \alpha \leq 1$.

a) Consider a POVM with two outcomes and respective measurement operators E and $\mathbb{1} - E$. Suppose that E has an eigenvalue λ such that $0 < \lambda < 1$. Show that the POVM is not extremal by expressing it as a nontrivial convex combination of two POVMs.
Hint: Consider the spectral decomposition of E and rewrite it as a convex combination of two POVM elements.

b) Suppose that E is an orthogonal projector. Show that the POVM cannot be expressed as a nontrivial convex combination of POVMs.

c) What is the operational interpretation of an element-wise convex combination of POVMs?

Exercise 7.3 Distance between channels

You have seen that TPCPMs may be used to define channels. In this exercise we will see some ways of quantifying how similar two channels are. Consider two TPCPMs

$$\mathcal{E}, \mathcal{F} : \mathcal{H}_A \mapsto \mathcal{H}_B. \tag{1}$$

One way of quantifying the distance between them is simply to see how distant two states transformed by those channels can be,

$$d(\mathcal{E}, \mathcal{F}) = \max_{\rho_A} \delta(\mathcal{E}(\rho_A), \mathcal{F}(\rho_A)), \tag{2}$$

where $\delta(\rho, \sigma)$ is the trace distance between states. However, we may want to consider that the state on which the channels act may be entangled with some other system, and therefore a channel that acts locally may produce global changes on the total state. The stabilized distance (Fig. 1) takes that into account:

$$\Delta(\mathcal{E}, \mathcal{F}) = \max_{\rho_{AR}} \delta(\mathcal{E} \otimes \mathcal{I}(\rho_{AR}), \mathcal{F} \otimes \mathcal{I}(\rho_{AR})), \tag{3}$$

where $\mathcal{I}$ is the identity map for operators that act on the reference system $\mathcal{H}_R$.

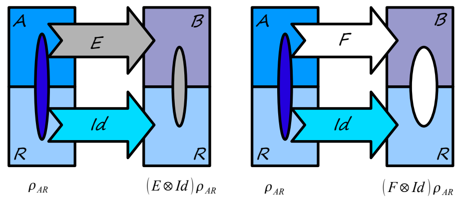

Figure 1: The stabilized distance between two maps considers that they act on a part of a larger entangled state.

a) Consider the fully depolarising channel on one qubit, $\mathcal{E}_p(\rho) = p\frac{\mathbb{1}}{2} + (1-p)\rho$, that can be expressed in the operator-sum representation ($\mathcal{E}(\rho) = \sum_k E_k \rho {E_k}^\dagger$) with the operators $\sqrt{1-\frac{3p}{4}}\mathbb{1}$ and $\frac{\sqrt{p}}{2}\sigma_i, \quad i = x, y, z$.
Compute and compare $d(\mathcal{E}_p, \mathcal{I})$ and $\Delta(\mathcal{E}_p, \mathcal{I})$.

b) Show that in general $d(\mathcal{E}, \mathcal{F}) \leq \Delta(\mathcal{E}, \mathcal{F})$.