Quantum Information Theory
Series 9

HS 14
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Exercise 9.1 Some properties of von Neumann entropy

We will now derive some properties of the von Neumann entropy that will be useful in later exercises. The von Neumann entropy of a density operator $\rho \in \mathcal{S}(\mathcal{H})$ is defined as

$$H(\rho) = -\mathrm{Tr}\big(\rho \log \rho\big) = -\sum_i \lambda_i \log \lambda_i, \tag{1}$$

where $\{\lambda_i\}_i$ are the eigenvalues of ρ.
Given a composite system $\rho_{ABC} \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$ and $\rho_{AB} = \mathrm{Tr}_C(\rho_{ABC})$ etc., we often write $H(AB)$ instead of $H(\rho_{AB})$ to denote the entropy of a subsystem.
The *conditional* von Neumann entropy may be defined in a composed system $\mathcal{H}_A \otimes \mathcal{H}_B$ as

$$H(A|B) = H(AB) - H(B). \tag{2}$$

The strong sub-additivity property of the von Neumann entropy proves very useful:

$$H(ABC) + H(B) \leq H(AB) + H(BC). \tag{3}$$

a) Prove the following general properties of the von Neumann entropy:

1. If ρ_{AB} is pure, then $H(A) = H(B)$.
2. If two subsystem are independent, $\rho_{AB} = \rho_A \otimes \rho_B$, then $H(AB) = H(A) + H(B)$.

b) Consider a bipartite system that is classical on a subsystem Z, namely $\rho_{ZA} = \sum_z p_z |z\rangle\langle z|_Z \otimes \rho_A^z$ for some basis $\{|z\rangle Z\}_z$ of $\mathcal{H}_Z$. Show that:

1. The entropy of the global state is given by

$$H(AZ) = H(Z) + \sum_z p_z\, H(A|Z = z), \tag{4}$$

where $H(A|Z = z) = H(\rho_A^z)$.

2. Even if one has access to subsystem A the classical variable is not fully known,

$$H(Z|A) \geq 0. \tag{5}$$

Remark: Eq (6) holds in general only for classical Z. Consider, e.g., the Bell-States as an immediate counterexample in the fully quantum case.

Exercise 9.2 Upper bound on von Neumann entropy

Given a state $\rho \in \mathcal{S}(\mathcal{H})$, show that

$$H(\rho) \leq \log |\mathcal{H}|. \tag{6}$$

Consider the state $\bar{\rho} = \int U \rho U^\dagger dU$, where the integral is over all unitaries $U \in \mathcal{U}(\mathcal{H})$ and dU is the Haar measure. Find $\bar{\rho}$ and use concavity of von Neumann entropy to show (7).
Hint: The Haar measure satisfies $d(UV) = d(VU) = dU$, where $V \in \mathcal{U}(\mathcal{H})$ is any unitary.

Exercise 9.3 Data Processing Inequality

Random variables X, Y, Z form a Markov chain $X \to Y \to Z$ if the conditional distribution of Z depends only on Y: $p(z|x,y) = p(z|y)$. The goal in this exercise is to prove the data processing inequality, $I(X : Y) \geq I(X : Z)$ for $X \to Y \to Z$.

1. First show the chain rule for mutual information: $I(X : YZ) = I(X : Z) + I(X : Y|Z)$, which holds for arbitrary X, Y, Z. The conditional mutual information is defined as

$$I(X : Y|Z) = \sum_z p(z) I(X : Y|Z = z) = \sum_z p(z) \sum_{x,y} p(x,y|z) \log \frac{p(x,y|z)}{p(x|z)p(y|z)}.$$

2. Next show that in a Markov chain $X \to Y \to Z$, X and Z are conditionally independent given Y; that is, $p(x,z|y) = p(x|y)p(z|y)$.

3. By expanding the mutual information $I(X : YZ)$ in two different ways, prove the data processing in equality.