Quantum Information Theory
Solutions 7

Exercise 7.1 Entropy properties

The von-Neumann entropy is defined as:

$$H(\rho) = -tr[\rho \log \rho] = -\sum_k \lambda_k \log \lambda_k$$

Show the following properties of this quantity:

1. $H(\rho) = 0$ iff ρ is pure
2. $H(U\rho U^*) = H(\rho)$ for unitary U
3. $H(\rho) \leq \log |supp\, \rho|$
4. $H(\sum_k p_k \rho_k) \geq \sum_k p_k H(\rho_k)$
5. $H(\sum_k P_k \rho P_k) \geq H(\rho)$ for any complete set of projectors P_k

Exercise 7.2 Geometry of Measurements

Let $F = \{F_1, F_2\}$ and $G = \{G_1, G_2\}$ be two POVMs. We define an element-wise convex combination of F and G as $\alpha F + (1-\alpha)G := \{\alpha F_1 + (1-\alpha)G_1, \alpha F_2 + (1-\alpha)G_2\}$, with $0 \leq \alpha \leq 1$.

a) *Consider a POVM with two outcomes and respective measurement operators E and $\mathbb{1} - E$. Suppose that E has an eigenvalue λ such that $0 < \lambda < 1$. Show that the POVM is not extremal by expressing it as a nontrivial convex combination of two POVMs.*

We expand E in its eigenbasis and write

$$\begin{aligned} E &= \lambda_0\ |0\rangle\langle 0| + \sum_{i\neq 0} \lambda_i\ |i\rangle\langle i| \\ &= \lambda_0\ |0\rangle\langle 0| + (1-\lambda_0)\ \sum_{i\neq 0} \lambda_i\ |i\rangle\langle i| + \lambda_0\ \sum_{i\neq 0} \lambda_i\ |i\rangle\langle i| \\ &= \lambda_0\ \underbrace{(|0\rangle\langle 0| + \sum_{i\neq 0} \lambda_i\ |i\rangle\langle i|)}_{E_1} + (1-\lambda_0) \underbrace{\sum_{i\neq 0} \lambda_i\ |i\rangle\langle i|}_{E_2}. \end{aligned}$$

Hence we can write the POVM $\{E, \mathbb{1} - E\}$ as a convex combination of the POVMs $\{E_1, \mathbb{1} - E_1\}$ and $\{E_2, \mathbb{1} - E_2\}$.

b) *Suppose that E is an orthogonal projector. Show that the POVM cannot be expressed as a nontrivial convex combination of POVMs.*

Let E be an orthogonal projector on some subspace $V \in \mathcal{H}$ and let $|\psi\rangle \in V^T$. If we assume that E can be written as the convex combination of two positive operators then

$$\begin{aligned} 0 &= \langle\psi|E|\psi\rangle \\ &= \lambda\langle\psi|E_1|\psi\rangle + (1-\lambda)\langle\psi|E_2|\psi\rangle. \end{aligned}$$

However, both terms on the right hand side are non-negative, thus they must vanish identically. Since $|\psi\rangle$ was arbitrary we conclude that $E_1 = E_2 = E$.

c) *What is the operational interpretation of an element-wise convex combination of POVMs?*

The element-wise convex combination of elements an be interpreted as using two different measurement devices with probability α and $1-\alpha$, but not knowing which measurement device was used. In contrast to that, a simple convex concatenation of sets would be interpreted as using two different measurement devices with probability α and $1-\alpha$, but keeping track of which measurement device was used. This is because by definition of a POVM, each POVM element corresponds to a specific measurement outcome. If the two POVMs are concatenated, we can still uniquely relate the measurement outcome to the corresponding measurement device.

The tips have more details and examples.

Exercise 7.3 Distance between channels

Consider two TPCPMs that define two channels,

$$\mathcal{E}, \mathcal{F} : \mathcal{H}_A \mapsto \mathcal{H}_B. \tag{1}$$

Let us call the naive *distance between channels the following quantity:*

$$d(\mathcal{E}, \mathcal{F}) = \max_{\rho_A} \delta(\mathcal{E}(\rho_A), \mathcal{F}(\rho_A)), \tag{2}$$

where $\delta(\rho, \sigma)$ is the trace distance between states. The stabilized distance between channels is defined as

$$d^{\diamond}(\mathcal{E}, \mathcal{F}) = \max_{\rho_{AR}} \delta(\mathcal{E} \otimes \mathcal{I}(\rho_{AR}), \mathcal{F} \otimes \mathcal{I}(\rho_{AR})), \tag{3}$$

where $\mathcal{I}$ is the identity map for operators that act on the reference system $\mathcal{H}_R$.

a) *Consider the fully depolarising channel on one qubit, $\mathcal{E}_p(\rho) = p\frac{\mathbb{1}}{2} + (1-p)\rho$, that can be expressed in the operator-sum representation ($\mathcal{E}(\rho) = \sum_k E_k \rho E_k{}^\dagger$) with the operators $\sqrt{1 - \frac{3p}{4}}\mathbb{1}$ and $\frac{\sqrt{p}}{2}\sigma_i, \quad i = x, y, z$.*

Compute and compare $d(\mathcal{E}_p, \mathcal{I})$ and $d^{\diamond}(\mathcal{E}_p, \mathcal{I})$.

There was a typo in the original version of the exercise: the channel acts as $\mathcal{E}_p(\rho) = p\frac{\mathbb{1}}{2} + (1-p)\rho$, and not as $\mathcal{E}_p(\rho) = p\mathbb{1} + (1-p)\rho$.

The distance $d(\mathcal{E}_p, \mathcal{I})$ is given by

$$\begin{aligned}
d(\mathcal{E}, \mathcal{I}) &= \max_{\rho} \delta(\mathcal{E}(\rho), \mathcal{I}(\rho)) \\
&= \max_{\rho} \frac{1}{2} \left| p\frac{\mathbb{1}}{2} + (1-p)\rho - \rho \right| \\
&= \max_{\rho} \frac{p}{2} \left| \frac{\mathbb{1}}{2} - \rho \right|,
\end{aligned}$$

which, if $\rho = \alpha|0\rangle\langle 0| + (1-\alpha)|1\rangle\langle 1|$ in its eigenbasis, is

$$\begin{aligned}
d(\mathcal{E}, \mathcal{I}) &= \max_{\rho} \frac{p}{2} \left(\left| \frac{1}{2} - \alpha \right| + \left| \frac{1}{2} - 1 + \alpha \right| \right), \qquad 0 \le \alpha \le 1 \\
&= \max_{\rho} p \left| \frac{1}{2} - \alpha \right|, \qquad 0 \le \alpha \le 1 \\
&= \frac{p}{2},
\end{aligned}$$

because $\left| \frac{1}{2} - \alpha \right|$ is maximised for pure states.

As for the diamond distance, we have

$$\begin{aligned}
d^{\diamond}(\mathcal{E}, \mathcal{I}) &= \max_{\rho_{AR}} \delta(\mathcal{E} \otimes \mathcal{I}(\rho_{AR}), \mathcal{I} \otimes \mathcal{I}(\rho_{AR})) \\
&= \max_{\rho_{AR}} \delta\left(\left(\sqrt{1 - \frac{3p}{4}}\mathbb{1} \otimes \mathbb{1} \right) \rho_{AR} \left(\sqrt{1 - \frac{3p}{4}}\mathbb{1} \otimes \mathbb{1} \right) + \sum_i \left[\left(\frac{\sqrt{p}}{2}\sigma_i \otimes \mathbb{1} \right) \rho_{AR} \left(\frac{\sqrt{p}}{2}\sigma_i \otimes \mathbb{1} \right) \right], \rho_{AR} \right) \\
&= \max_{\rho_{AR}} \delta\left(\left(1 - \frac{3p}{4} \right) \rho_{AR} + \frac{p}{4} \sum_i \left[(\sigma_i \otimes \mathbb{1}) \rho_{AR} (\sigma_i \otimes \mathbb{1}) \right], \rho_{AR} \right) \\
&= \max_{\rho_{AR}} \frac{1}{2} \left| \left(1 - \frac{3p}{4} \right) \rho_{AR} + \frac{p}{4} \sum_i \left[(\sigma_i \otimes \mathbb{1}) \rho_{AR} (\sigma_i \otimes \mathbb{1}) \right] - \rho_{AR} \right| \\
&= \max_{\rho_{AR}} \frac{p}{8} \left| \sum_i \left[(\sigma_i \otimes \mathbb{1}) \rho_{AR} (\sigma_i \otimes \mathbb{1}) \right] - 3\rho_{AR} \right|.
\end{aligned}$$

For now, instead of maximizing that quantity over all states we will apply it to the fully entangled state $|\Psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + |11\rangle$,

$$\begin{aligned} d^{\diamond}(\mathcal{E},\mathcal{I}) &\geq \frac{p}{8}\left|\sum_i \Big[(\sigma_i\otimes \mathbb{1})|\Psi\rangle\langle\Psi|_{AR}(\sigma_i\otimes\mathbb{1})\Big] - 3|\Psi\rangle\langle\Psi|_{AR}\right| \\ &= \frac{p}{8}\left|\begin{pmatrix} -1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & -1 \end{pmatrix}\right| = \frac{3p}{2}. \end{aligned}$$

Recall that the non-stabilized distance was only $d(\mathcal{E},\mathcal{I}) = \frac{p}{2}$ — it is possible to observe a gap between the stabilized and the non-stabilize distances. It can be shown that in fact $d^{\diamond}(\mathcal{E},\mathcal{I}) = \frac{3p}{2}$, i.e. the distance is optimized by a maximally entangled state.

b) *Show that in general* $d(\mathcal{E},\mathcal{F}) \leq d^{\diamond}(\mathcal{E},\mathcal{F})$.

We observe that, for any quantity X evaluated on states of $\mathcal{H}_A\otimes\mathcal{H}_R$,

$$\max_{\rho_{AR}} X(\rho_{AR}) \geq \max_{\rho_A} X\left(\rho_A\otimes\frac{\mathbb{1}_R}{|R|}\right),$$

as product states of the form $\rho_A\otimes\frac{\mathbb{1}_R}{|R|}$ are a subset of all quantum states of composed space $\mathcal{H}_A\otimes\mathcal{H}_R$.

Now we see that if we only consider these states, the two distances are equivalent, because

$$\begin{aligned} \delta(\mathcal{E}\otimes\mathcal{I}(\rho_A\otimes\frac{\mathbb{1}_R}{|R|}),\mathcal{F}\otimes\mathcal{I}(\rho_A\otimes\frac{\mathbb{1}_R}{|R|})) &= \frac{1}{2}\mathrm{Tr}\left(\mathcal{E}(\rho_A)\otimes\frac{\mathbb{1}_R}{|R|} - \mathcal{F}(\rho_A)\otimes\frac{\mathbb{1}_R}{|R|}\right) \\ &= \frac{1}{2}\mathrm{Tr}\left([\mathcal{E}(\rho_A)-\mathcal{F}(\rho_A)]\otimes\frac{\mathbb{1}_R}{|R|}\right) \qquad ^{(*)} \\ &= \frac{1}{2}\mathrm{Tr}\left(\mathcal{E}(\rho_A)-\mathcal{F}(\rho_A)\right), \qquad ^{(**)} \end{aligned}$$

where $^{(*)}$ stands because $A\otimes C + B\otimes C = [A+B]\otimes C$ and $^{(**)}$ because $\mathrm{Tr}(A\otimes\mathbb{1}_R) = |R|\ \mathrm{Tr}(A)$. Putting everything together, we have

$$\begin{aligned} d^{\diamond}(\mathcal{E},\mathcal{F}) &= \max_{\rho_{AR}\in\mathcal{S}(\mathcal{H}_A\otimes\mathcal{H}_R)} \delta(\mathcal{E}\otimes\mathcal{I}(\rho_{AR}),\mathcal{F}\otimes\mathcal{I}(\rho_{AR})) \\ &\geq \max_{\rho_{AR}\in\mathcal{W}\subset\mathcal{S}(\mathcal{H}_A\otimes\mathcal{H}_R)} \delta(\mathcal{E}\otimes\mathcal{I}(\rho_{AR}),\mathcal{F}\otimes\mathcal{I}(\rho_{AR})) \\ &= \max_{\rho_A\in\mathcal{S}(\mathcal{H}_A)} \delta(\mathcal{E}\otimes\mathcal{I}(\rho_A\otimes\frac{\mathbb{1}_R}{|R|}),\mathcal{F}\otimes\mathcal{I}(\rho_A\otimes\frac{\mathbb{1}_R}{|R|})) \\ &= \max_{\rho_A\in\mathcal{S}(\mathcal{H}_A)} \delta(\mathcal{E}(\rho_A),\mathcal{F}(\rho_A)) \\ &= d(\mathcal{E},\mathcal{F}). \end{aligned}$$