
Exercise 2.1 Bloch sphere

In this exercise we will see how we may represent qubit states as points in a three-dimensional unit ball.
A qubit is a two level system, whose Hilbert space is equivalent to $\mathbb{C}^2$. The Pauli matrices together with the identity form a basis for 2×2 Hermitian matrices,

$$\mathcal{B} = \left\{ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}, \tag{1}$$

where the matrices are represented in basis $\{|0\rangle, |1\rangle\}$.
We will see that density operators can always be expressed as

$$\rho = \frac{1}{2}(\mathbb{1} + \vec{r} \cdot \vec{\sigma}) \tag{2}$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ and $\vec{r} = (r_x, r_y, r_z), |\vec{r}| \leq 1$ is the so-called Bloch vector, that gives us the position of a point in an unit ball. The surface of that ball is usually known as the Bloch sphere.

a) *Show that the Pauli matrices respect following commutation relations:*

$$[\sigma_i, \sigma_j] := \sigma_i\sigma_j - \sigma_j\sigma_i = 2\varepsilon_{ijk}\sigma_k, \tag{3}$$
$$\{\sigma_i, \sigma_j\} := \sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}\mathbb{1}. \tag{4}$$

This is trivially obtained using linear algebra and applying the definitions of the Pauli matrices.

b) *Show that the operator ρ defined in (2) is a valid density operator for any vector $\vec{r}$ with $|\vec{r}| \leq 1$ by proving it fulfills the following properties:*

1) *Hermiticity: $\rho = \rho^\dagger$.*
   All Pauli matrices are Hermitian and the vector $\vec{r}$ is real, so the result comes from direct application of (2).
2) *Positivity: $\rho \geq 0$.* First we will show that a density matrix is positive if and only if it self-adjoint has non-negative eigenvalues.
   $\Rightarrow$: If ρ is positive, this means that for any $|\psi\rangle \in H$, $\langle\psi, \rho\psi\rangle \geq 0$. Hence $\langle\psi, \rho\psi\rangle$ is also real. So we have that:

   $$\langle\psi, \rho\psi\rangle = \langle\rho\psi, \psi\rangle = \langle\psi, \rho^*\psi\rangle$$

   Now using this equality and the polarisation identity

   $$\langle\psi, \phi\rangle = \sum_{k=0}^{3} i^{-k}\langle\psi + i^k\phi, \phi + i^k\psi\rangle$$

   we have that $\langle\psi, \rho\phi\rangle = \langle\rho\psi, \phi\rangle$, for any ψ and ϕ, hence ρ is self-adjoint.

To show that eigenvalues must be non-negative, we look at a normalised eigenvector ψ of ρ s.t. $\rho\psi = \lambda\psi$. Then

$$\langle \psi, \rho\psi \rangle = \langle \psi, \lambda\psi \rangle = \lambda \left|\psi\right|^2 \geq 0$$

using the definition of the positivity of ρ. Hence this holds for each eigenvalue, and all of them must be non-negative.

$\Leftarrow$: Now assume ρ is self-adjoint and has non-negative eigenvalues. Choose an orthonormal basis ϕ_j of eigenvectors, i.e. $\rho\phi_j = \lambda_j \phi_j$. Any vector $\phi \in H$, we can expand as $\phi = \sum_j c_j \phi_j$, for some coefficients c_j. Hence

$$\begin{gathered}\langle \phi, \rho\phi \rangle = \\ \sum_{j,k} \overline{c_j} c_k \langle \phi_j, \rho\phi_k \rangle = \\ \sum_{j,k} \overline{c_j} c_k \langle \phi_j, \lambda_k \phi_k \rangle = \\ \sum_{j,k} \overline{c_j} c_k \lambda_k \langle \phi_j, \phi_k \rangle = \\ \sum_j \left|c_j\right|^2 \lambda_j \geq 0\end{gathered} \tag{5}$$

So ρ is positive operator. With this we end the proof. Hence, as we know that ρ is self-adjoint, for positivity it remains to prove that it has non-negative eigenvalues.

The general form of a state given by (2) is

$$\rho = \frac{1}{2}\begin{pmatrix} 1 + r_z & r_x - ir_y \\ r_x + ir_y & 1 - r_z \end{pmatrix} \quad \Rightarrow \quad \text{Eigenvalues: } \left\{\frac{1 - |\vec{r}|}{2}, \frac{1 + |\vec{r}|}{2}\right\}. \tag{6}$$

Since $0 \leq |\vec{r}| \leq 1$, the eigenvalues are non-negative. From previous part we also know that ρ is self-adjoint. Hence ρ is a positive matrix (operator).

3) *Normalisation: Tr*$(\rho) = 1$.
From (6) we have that

$$\text{Tr}(\rho) = \frac{1 - |\vec{r}|}{2} + \frac{1 + |\vec{r}|}{2} = 1.$$

c) *Now do the converse: show that any two-level density operator may be written as* (2).

We show this in a matrix formalism.

We represent ρ as 2×2 matrix, and as we know that it is self-adjoint, we can write it as:

$$\rho = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

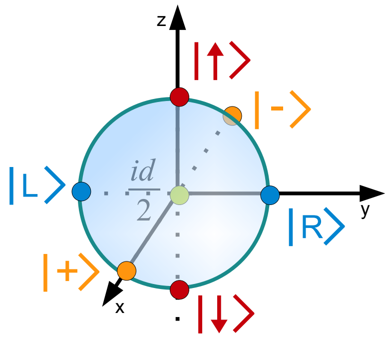
with α and δ real, and $\beta = \overline{\gamma}$, $\alpha + \delta = 1$ for unit trace. If we introduce $\alpha = \frac{1}{2}(1 + r_3), \delta = \frac{1}{2}(1 - r_3), \gamma = \frac{1}{2}(r_1 + ir_2)$, we have $\rho = \frac{1}{2}(1 + \vec{r} \cdot \vec{\sigma})$. Now ρ also has to have non-negative eigenvalues (since it is a positive operator), hence $det(\rho)$ must be non-negative $\Rightarrow 1 - |\vec{r}|^2 \geq 0 \Rightarrow |\vec{r}|^2 \leq 1$.

d) *Check that the surface of the ball — the Bloch sphere — is formed by all the pure states.*

If ρ defines a pure state then $\rho = |\psi\rangle\langle\psi|$ for some vector ψ. Hence $\rho\psi = \psi$ and $\rho\phi = 0$ for ϕ orthogonal to ψ (ϕ and ψ are eigenvectors of ρ). Hence eigenvalues are 1 and 0, so $det(\rho) = 0$ and we have $|\vec{r}| = 1$.

e) *Find and draw in the ball the Bloch vectors of a fully mixed state and the pure states that form three bases,* $\{|\uparrow\rangle, |\downarrow\rangle\}$, $\{|+\rangle, |-\rangle\}$ *and* $\{|\circlearrowright\rangle, |\circlearrowleft\rangle\}$. *Hint: Use* $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$ *and* $|\circlearrowleft / \circlearrowright\rangle = (|0\rangle \pm i|1\rangle)/\sqrt{2}$.

state	**density matrix**	**Bloch vector**	**in the figure**
$\frac{\mathbb{1}}{2}$	$\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$(0,0,0)$	green
$\lvert 0\rangle$	$\frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$	$(0,0,1)$	red
$\lvert 1\rangle$	$\frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$	$(0,0,-1)$	red
$\lvert +\rangle$	$\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$	$(1,0,0)$	yellow
$\lvert -\rangle$	$\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$	$(-1,0,0)$	yellow
$\lvert \circlearrowright\rangle$	$\frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$	$(0,1,0)$	blue: $\lvert R\rangle$
$\lvert \circlearrowleft\rangle$	$\frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$	$(0,-1,0)$	blue: $\lvert L\rangle$



f) *Find and diagonalise the states represented by Bloch vectors* $\vec{r}_1 = (\frac{1}{2}, 0, 0)$ *and* $\vec{r}_2 = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$.

We have

$$\begin{aligned}
\rho_1 &= \frac{1}{2}\left[\mathbb{1} + \left(\frac{1}{2}, 0, 0\right) \cdot (\sigma_x, \sigma_y, \sigma_z)\right] \\
&= \frac{1}{2}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] \\
&= \frac{1}{4}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \qquad \Rightarrow \qquad \text{Eigenvalues: } \left\{\frac{1}{4}, \frac{3}{4}\right\},
\end{aligned}$$

$$\begin{aligned}
\rho_2 &= \frac{1}{2}\left[\mathbb{1} + \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) \cdot (\sigma_x, \sigma_y, \sigma_z)\right] \\
&= \frac{1}{2}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] \\
&= \frac{1}{2\sqrt{2}}\begin{pmatrix} \sqrt{2}+1 & 1 \\ 1 & \sqrt{2}-1 \end{pmatrix} \qquad \Rightarrow \qquad \text{Eigenvalues: } \{0, 1\}.
\end{aligned}$$

The first Bloch vector lies inside the ball ($|\vec{r}_1 = \frac{1}{4}|$), and the state that it represents is mixed. The Bloch vector of the second state is on the surface of the sphere, and that state is pure.

Exercise 2.2 The Hadamard Gate

An important qubit transformation in quantum information theory is the Hadamard gate. In the basis of $\sigma_{\hat{z}}$, it takes the form

$$H = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} . \tag{7}$$

That is to say, if $|0\rangle$ and $|1\rangle$ are the $\sigma_{\hat{z}}$ eigenstates, corresponding to eigenvalues $+1$ and -1, respectively, then

$$H = \tfrac{1}{\sqrt{2}} \left(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1| \right) \tag{8}$$

1. *Show that H is unitary.*

   A matrix U is unitary when $U^\dagger U = \mathbb{1}$. In fact, $H^\dagger = H$, so we just need to verify that $H^2 = \mathbb{1}$, which is the case.

2. *What are the eigenvalues and eigenvectors of H?*

   Since $H^2 = \mathbb{1}$, its eigenvalues must be ± 1. If both eigenvalues were equal, it would be proportional to the identity matrix. Thus, one eigenvalue is $+1$ and the other -1. By direct calculation we can find that the (normalized) eigenvectors are

$$|\lambda_\pm\rangle = \pm \frac{\sqrt{2 \pm \sqrt{2}}}{2} |0\rangle + \frac{1}{\sqrt{2(2 \pm \sqrt{2})}} |1\rangle \tag{9}$$

3. *What form does H take in the basis of $\sigma_{\hat{x}}$? $\sigma_{\hat{y}}$?*

   The eigenbasis of $\sigma_{\hat{x}}$ is formed by the two states $|\hat{x}_\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$. From the form of H given in (8), it is clear that we can express H as

$$H = |\hat{x}_+\rangle\langle 0| + |\hat{x}_-\rangle\langle 1| \quad \text{or} \tag{10}$$

$$H = |0\rangle\langle \hat{x}_+| + |1\rangle\langle \hat{x}_-| \tag{11}$$

   The latter form follows immediately from the first since $H^\dagger = H$. Finally, we can express the $\sigma_{\hat{z}}$ basis $|0/1\rangle$ in terms of the $\sigma_{\hat{x}}$ basis as $|0\rangle = \frac{1}{\sqrt{2}}(|\hat{x}_+\rangle + |\hat{x}_-\rangle)$ and $|1\rangle = \frac{1}{\sqrt{2}}(|\hat{x}_+\rangle - |\hat{x}_-\rangle)$. Thus, if we replace $|0\rangle$ and $|1\rangle$ by these expressions in the equation for H we find

$$H = |0\rangle\langle \hat{x}_+| + |1\rangle\langle \hat{x}_-| = \tfrac{1}{\sqrt{2}} \left(|\hat{x}_+\rangle\langle \hat{x}_+| + |\hat{x}_-\rangle\langle \hat{x}_+| + |\hat{x}_+\rangle\langle \hat{x}_-| - |\hat{x}_-\rangle\langle \hat{x}_-| \right) . \tag{12}$$

   Evidently, H has exactly the same representation in the $\sigma_{\hat{x}}$ basis! In retrospect, we should have anticipated this immediately once we noticed that H interchanges the $\sigma_{\hat{z}}$ and $\sigma_{\hat{x}}$ bases.

   For $\sigma_{\hat{y}}$, we can proceed differently. What is the action of H on the $\sigma_{\hat{y}}$ eigenstates? These are $|\hat{y}_\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm i|1\rangle)$. Thus,

$$
\begin{aligned}
H|\hat{y}_\pm\rangle &= \frac{1}{\sqrt{2}}\left(H|0\rangle \pm iH|1\rangle\right) && (13)\\
&= \frac{1}{2}\left(|0\rangle + |1\rangle \pm i|0\rangle \mp i|1\rangle\right) && (14)\\
&= \left(\frac{1 \pm i}{2}\right)|0\rangle + \left(\frac{1 \mp i}{2}\right)|1\rangle && (15)\\
&= \frac{1}{\sqrt{2}}e^{i \pm \frac{\pi}{4}}\left(|0\rangle + \left(\frac{1 \mp i}{1 \pm i}\right)|1\rangle\right) && (16)\\
&= \frac{1}{\sqrt{2}}e^{i \pm \frac{\pi}{4}}\left(|0\rangle \mp i|1\rangle\right) && (17)\\
&= e^{i \pm \frac{\pi}{4}}|\hat{y}_\mp\rangle && (18)
\end{aligned}
$$

Therefore, the Hadamard operation just swaps the two states in the basis (note that if we used a different phase convention for defining the $\sigma_{\hat{y}}$ eigenstates, there would be extra phase factors in this equation). So, $H = \begin{pmatrix} 0 & e^{-i\frac{\pi}{4}} \\ e^{i\frac{\pi}{4}} & 0 \end{pmatrix}$ in this basis.

4. *Give a geometric interpretation of the action of H in terms of the Bloch sphere.*

   All unitary operators on a qubit are rotations of the Bloch sphere by some angle about some axis. Since $H^2 = \mathbb{1}$, it must be a π rotation. Because the $\hat{y}$-axis is interchanged under H, the axis must lie somewhere in the $\hat{x}$-$\hat{z}$ plane. Finally, since H interchanges the $\sigma_{\hat{x}}$ and $\sigma_{\hat{z}}$ bases, it must be a rotation about the $\hat{m} = \frac{1}{\sqrt{2}}(\hat{x} + \hat{z})$ axis.

   Easier way to see this, is by observing that $H = \frac{1}{\sqrt{2}}(\sigma_x + \sigma_z)$. As Pauli matrices represent rotations for an angle π around the suitable axis (σ_x around x-axis etc.), with some help of geometry and algebra we can see that H represents a π rotation (as $H^2 = \mathbb{1}$) around the axis $\frac{1}{\sqrt{2}}(\hat{x} + \hat{z})$.