Introduction to Superconductivity
Exercise Sheet 4

Discussion date: 15 October 2014

Exercise 1: Ising model, mean-field theory and Ginzburg-Landau formalism.

The concept of spontaneous symmetry breaking is illustrated for the Ising model. You revise the model and how to solve it using mean-field theory, and we introduce the Ginzburg-Landau formalism.

We consider the Ising model for N spins $\sigma_i = \pm 1$, where nearest-neighbors $\langle i, j\rangle$ have an interaction J_{ij}, and including a magnetic field h_i. The Hamiltonian is given by

$$\mathcal{H} = -\sum_{\langle i,j\rangle} J_{ij}\sigma_i\sigma_j - \sum_i h_i\sigma_i. \tag{1}$$

$\boxed{1}$ First, we consider **free spins** without any interaction:

$$J_{ij} = 0, \quad h_i = 0. \tag{2}$$

(a) What is the expectation value for a single spin, $m = \langle\sigma_i\rangle$? What is $\langle\sigma_i^2\rangle - \langle\sigma_i\rangle^2$?

(b) For the average magnetization $M = \frac{1}{N}\sum_i \sigma_i$, what is $\langle M\rangle$? What is $\langle M^2\rangle - \langle M\rangle^2$?

$\boxed{2}$ Next, we include a **constant ferromagnetic** nearest-neighbor interaction:

$$J_{ij} = J > 0. \tag{3}$$

(c) Now, what is $\langle\sigma_i\rangle$? What is $\langle\sigma_i^2\rangle - \langle\sigma_i\rangle^2$?

(d) What is $\langle M\rangle$?

Comment: The calculation and discussion of the correlation $\langle\sigma_i\sigma_j\rangle$ is more involved. We will do this together in the exercise class.

$\boxed{3}$ Finally, we include a **constant magnetic field**:

$$h_i = H. \tag{4}$$

(e) Using the mean-field approach, find the self-consistency equation for $m = \langle\sigma_i\rangle$.
Analyze the self-consistency equation graphically. What is the critical temperature T_c below which a magnetization $m \neq 0$ is possible even for $H = 0$?

(f) Find the free energy $F(T, H, m) = -k_B T \ln Z_N$.
Check: The equilibrium condition $\frac{\partial F}{\partial m} = 0$ reproduces the self-consistency equation.

(g) Which of the three solutions below T_c actually minimizes the free energy? Expand F first for small m and H (small h_{eff}), and then additionally at $T \approx T_c$. Plot $F(m)$ for the different temperature regimes at $H = 0$ and discuss.
Comment: This is the Landau expansion.

(h) Find an expression for $m(T)$ by minimizing the approximated $F(m)$, and plot your solution.