Goldstone modes and the Anderson Higgs mechanism.

If we consider a charge neutral system with order.

$$\langle \psi \rangle = \psi_0 e^{i\phi_0} \qquad \psi_0 = \sqrt{\frac{-\alpha}{2\beta}}.$$

$$\& \quad F = \int_x \left[\gamma \nabla\psi^* \nabla\psi + \alpha |\psi|^2 + \frac{\beta}{2} |\psi|^4 \right] \quad (1)$$

Then fluctuations around the order are defined by

$$\psi(r) = \left[|\psi_0| + \delta\psi \right] e^{i[\phi_0 + \delta\phi]} \quad (2)$$

$\delta\psi, \delta\phi \rightarrow$ real fns. describing phase and amplitude fluctuations.

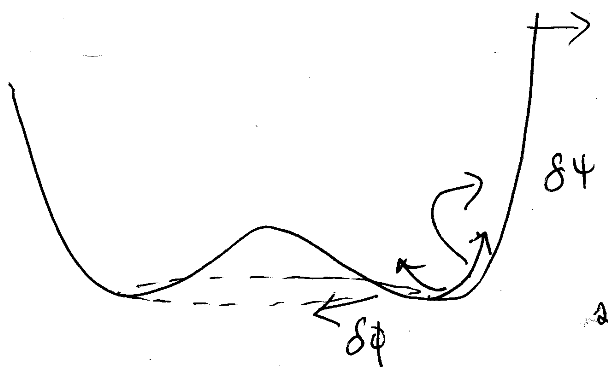
(2) $\rightarrow$ (1)

$$\therefore \delta F = \int_x \gamma (\nabla \delta\psi)^2 + 2|\alpha| \delta\psi^2 + |\psi_0|^2 \gamma (\nabla \delta\phi)^2 + \ldots$$

$\delta\psi \rightarrow$ amplitude flns. cost energy [massive mode $m = 2|\alpha|$]

$\delta\phi \rightarrow$ phase fluctuations are smooth and do not cost much. [massless mode $\rightarrow m = 0$!]

$\rightarrow \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4.$



Phase flns. relate a degenerate manifold of ground states $\rightarrow$ implies existence of low energy excitations called Goldstone modes.

In general spontaneous symmetry breaking
$\Longrightarrow$ existence of such low energy Goldstone modes.

Goldstone theorem:

Refs. Y. Nambu, Phys. Rev. 117, 648 (1960)
J. Goldstone et al, Phys Rev. 127, 965.
Also in various textbooks on Particle Physics.

What about the Superconductor?

Note: In non-relativistic superfluids, amplitude modes are not really gapped! [Beyond the level and context of course].

free energy density: $F = \int_x f$.

~~ftotal~~ $f = f_0 + \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{1}{2m}\left|\left[\frac{\hbar\vec{\nabla}}{i} - q\vec{A}\right]\psi\right|^2 + \frac{(\vec{\nabla}\times\vec{A})^2}{2\mu_0}$

Focus only on phase fluctuations as phase invariance is broken in SC state [global U(1) symmetry].

Below T_c, assume $\langle\psi\rangle = \psi_0\, e^{i[\theta(x) + \phi_0]}$

then,

$$F = f_0 + f_S^{mt} + \int_x \underbrace{\frac{|\psi_0|^2}{2m}\left[\hbar\vec{\nabla}\theta - q\vec{A}\right]^2}_{} + \frac{(\vec{\nabla}\times\vec{A})^2}{2\mu_0}.$$

If $\vec{A} = 0$, then can be rewritten in Fourier space as

$$f \propto \int_x \frac{\hbar^2|\psi_0|^2(\vec{\nabla}\theta)^2}{2m} \equiv \sum_{\vec{p}} \frac{\hbar^2|\psi_0|^2|\vec{p}|^2}{2m}\,\theta_{\vec{p}}\,\theta_{-\vec{p}}$$

Since in Superconductors we deal with charged particles we cannot neglect electrodynamics.

The free energy

$$f = F[\theta, \vec{A}] = f_0 + f_s + \sum_{\vec{p}} \Big\{ \frac{\hbar^2 |\psi_0|^2}{2m} \left[i\vec{p}\,\theta_{\vec{p}} - \frac{q}{\hbar}\vec{A}_p \right] \left[-i\vec{p}\,\theta_{-\vec{p}} - \frac{q}{\hbar}\vec{A}_{-p} \right] + \frac{1}{2\mu_0} (i\vec{p} \times \vec{A}_p)(-i\vec{p} \times \vec{A}_{-\vec{p}}) \Big\}.$$

$$e^{-\frac{\mathcal{F}[\vec{A}]}{k_B T}} = \int \mathcal{D}\theta\, e^{-F[\theta, \vec{A}]/k_B T}.$$

$\mathcal{D}\theta \rightarrow$ integral measure over all configurations of θ.

$$\mathcal{D}\theta = \prod_{\{x\}} d\theta(x).$$

$$F = f_s + f_0 + \sum_{\vec{p}} \Big[\frac{\hbar^2 |\psi_0|^2}{2m} \Big\{ |\vec{p}|^2 \theta_{\vec{p}}\, \theta_{-\vec{p}} + \frac{q^2}{\hbar^2} \vec{A}_{\vec{p}} \cdot \vec{A}_{-\vec{p}} - \frac{q}{\hbar} (i\vec{p}\, \vec{A}_{\vec{p}}\, \theta_{\vec{p}} - i\vec{p} \cdot \vec{A}_p\, \theta_{-\vec{p}}) \Big\} + \frac{1}{2\mu_0} (\vec{p} \times \vec{A}^{\perp}_{\vec{p}})(\vec{p} \times \vec{A}^{\perp}_{-\vec{p}}) \Big]$$

We have introduced, $\vec{A}_{\vec{p}} = \vec{A}^{\parallel}_{\vec{p}} + \vec{A}^{\perp}_{\vec{p}}$

Define, $\tilde{\theta}_p = \theta_p + \frac{iq}{\hbar} \frac{\vec{p} \cdot \vec{A}_{\vec{p}}}{|\vec{p}|^2}$

then $$F = f_s + f_0 + \sum_{\vec{p}} \frac{\hbar^2 |\psi_0|^2}{2m} \Big[|\vec{p}|^2 \tilde{\theta}_p\, \tilde{\theta}_{-p} + \frac{q^2}{\hbar^2} \vec{A}_p \cdot \vec{A}_{-p}$$

$$- \frac{|\psi_0|^2}{2m} \frac{q^2}{|\vec{p}|^2} (\vec{p}\cdot\vec{A}_p)(\vec{p}\cdot\vec{A}_{-p}) + \frac{1}{2\mu_0} (\vec{p}\times\vec{A}_p^{\perp})(\vec{p}\times\vec{A}_{-p}^{\perp}) \Big) \Big]$$

The shift in θ_p to $\tilde{\theta}_p$ does not affect the integral because the measure samples all configurations

[Just like $\int_{-\infty}^{\infty} e^{-y^2} dy = \int_{-\infty}^{\infty} e^{-(x+a)^2} dx$.]

$\therefore$ we can integrate over $\tilde{\theta}_p$ to obtain

$$f(A) \;\propto\; \sum_{\vec{p}} \frac{1}{2\mu_0} \Big[\underbrace{|\vec{p}|^2 + \frac{\hbar^2 |\psi_0|^2}{\cancel{2}m} \cancel{2}\mu_0 \frac{q^2}{\hbar^2}} \Big] \vec{A}^{\perp}_{\vec{p}} \cdot \vec{A}^{\perp}_{-\vec{p}}$$

$$\downarrow$$

$$[\,|\vec{p}|^2 + \lambda_L^{-2}\,]$$

$\lambda_L \rightarrow$ London penetration depth. $\lambda_L^2 = \dfrac{m}{\mu_0 q^2 |\psi_0|^2}$

Reminiscent of $E^2(\vec{p}) = \vec{p}^{\,2} c^2 + m^2 c^4$.

relativistic dispersion of a massive particle.

$\Rightarrow$ Gauge field absorbs the Goldstone mode & becomes massive $\Rightarrow$ photon has mass

[$\Rightarrow$ Meissner effect].

$\longrightarrow$ ANDERSON - HIGGS MECHANISM

Seen in particle physics too.

The analogous Higgs mode is related to amplitude fluctuations and is similar to the Higgs boson of Electroweak Theory.

The real derivation of the Higgs mode is very complex within Ginzburg Landau formalism and is beyond the scope of the course.

But look at the perspective article in Science mentioned on website