Exercise 1. ***Playing around with wave functions in second quantization.***

In the formalism of second quantization, a general state of N particles at positions $\vec{r}_1$, $\vec{r}_2$, ... is given by

$$|\vec{r}_1, \vec{r}_2, ..., \vec{r}_N\rangle = \frac{1}{\sqrt{N!}} \hat{\Psi}^\dagger (r_N) \cdots \hat{\Psi}^\dagger (r_1) |0\rangle \ , \tag{1}$$

where $|0\rangle$ is the vacuum state and the field operators $\hat{\Psi}(\vec{r})$ are defined as

$$\hat{\Psi}(\vec{r}) = \sum_k \phi_k(\vec{r})\ \hat{a}_k \ , \tag{2}$$

with $\hat{a}_k$ the annihilator of mode k and $\phi_k(\vec{r})$ the one-particle wave function of mode k.

Consider a state $|\psi\rangle$ of three particles in modes k_1, k_2, and k_3. Consider its wave function

$$\psi(\vec{r}_1, \vec{r}_2, \vec{r}_3) = \langle \vec{r}_1, \vec{r}_2, \vec{r}_3 | \psi\rangle = \langle \vec{r}_1, \vec{r}_2, \vec{r}_3 | \hat{a}^\dagger_{k_3} \hat{a}^\dagger_{k_2} \hat{a}^\dagger_{k_1} |0\rangle \ . \tag{3}$$

(a) First calculate the vacuum expectation value

$$\langle 0 | \hat{a}_{\ell_1} \hat{a}_{\ell_2} \hat{a}_{\ell_3} \hat{a}^\dagger_{k_3} \hat{a}^\dagger_{k_2} \hat{a}^\dagger_{k_1} |0\rangle \ , \tag{4}$$

for bosons and for fermions.

(b) Determine $\psi(\vec{r}_1, \vec{r}_2, \vec{r}_3)$ for bosons and for fermions. What symmetries does the wave function possess?

(c) Determine the normalization of the wave function for fermions and for bosons. First consider the case where k_1, k_2 and k_3 are all different, and then study the case where two or more modes are the same. What do you observe?

Note: for the lazy, it is also possible to do the whole exercise with two particles only. For the motivated, calculate it for N particles.

Exercise 2. ***Correlation functions in 1D***

Consider particles of mass m in 1 dimension that sit on a ring of (very large) length L (this is a system with periodic boundary conditions).

(a) Calculate the pair correlation function for fermions in the cases $T = 0$ and in the limit of high temperature. How do the resulting functions differ from the 3 dimensional case? Try to give a physical interpretation and explanation for what you find.

(b) Do the same for bosons, distinguishing the low temperature case (condensate) and the high temperature limit. Again, comment on the differences to the corresponding results in $3D$ and their relations to the fermionic case.

Office Hours: Monday, November 17, 8-10 AM (Lea Krämer, HIT K 32.2).