Exercise 1. ***Magnetic domain wall.***

We want to calculate the energy of a magnetic domain wall in the framework of the Ginzburg-Landau (GL) theory. Assuming translational symmetry in the (y, z)-plane, the GL functional in zero field reads

$$F[m, m'] = F_0 + \int dx \left\{ \frac{A}{2} m(x)^2 + \frac{B}{4} m(x)^4 + \frac{\kappa}{2} [m'(x)]^2 \right\} . \tag{1}$$

(a) Solve the GL equation with boundary conditions

$$m(x \to \pm\infty) = \pm m_0, \quad m'(x \to \pm\infty) = 0, \tag{2}$$

where m_0 is the magnetization of the uniform solution.

(b) First, find the energy of the uniformly polarized solution (no domain walls). Next, compute the energy of the solution with a domain wall compared to the uniform solution. Use the coefficients A, B and κ according to the expansion of the mean-field free energy of the Ising model (see Eqs. (5.78) and (5.83)). Finally, find the energy of a sharp step in the magnetization and compare it to the above results.

Exercise 2. ***Temperature dependence of the superfluid fraction.***

In the lecture we calculated the number of condensed (superfluid) particles at zero temperature [Eq. (7.31)]. In this exercise we want to determine the temperature dependence of this fraction in the limit $T \to 0$.

(a) Calculate the expectation value of the density of particles with momentum $\boldsymbol{k}$,

$$n_{\boldsymbol{k}} := \frac{1}{\Omega} \left\langle \hat{a}^\dagger_{\boldsymbol{k}} \hat{a}_{\boldsymbol{k}} \right\rangle . \tag{3}$$

(b) Approximate the temperature dependence of this density,

$$\delta n_{\boldsymbol{k}}(T) := n_{\boldsymbol{k}}(T) - n_{\boldsymbol{k}}(T = 0) , \tag{4}$$

in the limit $T \to 0$.

(c) Calculate the temperature dependence of the density of condensed particles,

$$\delta n_0 = - \sum_{\boldsymbol{k}} \delta n_{\boldsymbol{k}} , \tag{5}$$

in the same limit. What happens in a two-dimensional system?

Hint. Keep only the terms of lowest order in T.

(d) Calculate the expectation value $\left\langle \hat{a}^\dagger_{\boldsymbol{k}} \hat{a}^\dagger_{-\boldsymbol{k}} \right\rangle$. What is the physical interpretation of this quantity?

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