Ideal Quantum Gases

independent particles: plane wave states $|\vec{p}\rangle$ with $\epsilon_{\vec{p}} = \frac{\vec{p}^2}{2m}$

Fermions: Pauli prinziple, i.e. at most one particle per state

Bosons: unlimited number of particles per state

grand canonical ensemble for practical reasons

$$\mathcal{Z} = \prod_{\vec{p}} \sum_{n_{\vec{p}}} \left(z e^{-\beta \epsilon_{\vec{p}}}\right)^{n_{\vec{p}}} = \begin{cases} \prod_{\vec{p}} \left(1 + z e^{-\beta \epsilon_{\vec{p}}}\right) & \text{Fermions} \quad n_{\vec{p}} = 0, 1 \\ \prod_{\vec{p}} \frac{1}{1 - z e^{-\beta \epsilon_{\vec{p}}}} & \text{Bosons} \quad n_{\vec{p}} = 0, 1, 2, \ldots \end{cases}$$

geometric series

fugacity: $z = e^{\beta \mu}$

Ideal Quantum Gases

equations of state

$$\frac{pV}{k_B T} = -\beta\Omega(T,V,\mu) = \ln \mathcal{Z} = \begin{cases} \sum_{\vec{p}} \ln\left(1 + z e^{-\beta\epsilon_{\vec{p}}}\right) & \text{Fermions} \\ -\sum_{\vec{p}} \ln\left(1 - z e^{-\beta\epsilon_{\vec{p}}}\right) & \text{Bosons} \end{cases}$$

note: general relation for mono-atomic ideal gases $U = \frac{3}{2} pV$

particle number

$$N = z\frac{\partial}{\partial z}\ln \mathcal{Z} = \begin{cases} \sum_{\vec{p}} \frac{1}{e^{\beta\epsilon_{\vec{p}}} z^{-1} + 1} & \text{Fermions} \\ \sum_{\vec{p}} \frac{1}{e^{\beta\epsilon_{\vec{p}}} z^{-1} - 1} & \text{Bosons} \end{cases}$$

$$\langle n_{\vec{p}} \rangle = \frac{1}{z^{-1} e^{\beta\epsilon_{\vec{p}}} + 1}$$

Fermi-Dirac distribution

$$\langle n_{\vec{p}} \rangle = \frac{1}{z^{-1} e^{\beta\epsilon_{\vec{p}}} - 1}$$

Bose-Einstein distribution

$$\frac{p}{k_B T} = \frac{2s+1}{\lambda^3} f_{5/2}(z)$$

$$\frac{1}{v} = \frac{N}{V} = \frac{2s+1}{\lambda^3} f_{3/2}(z)$$

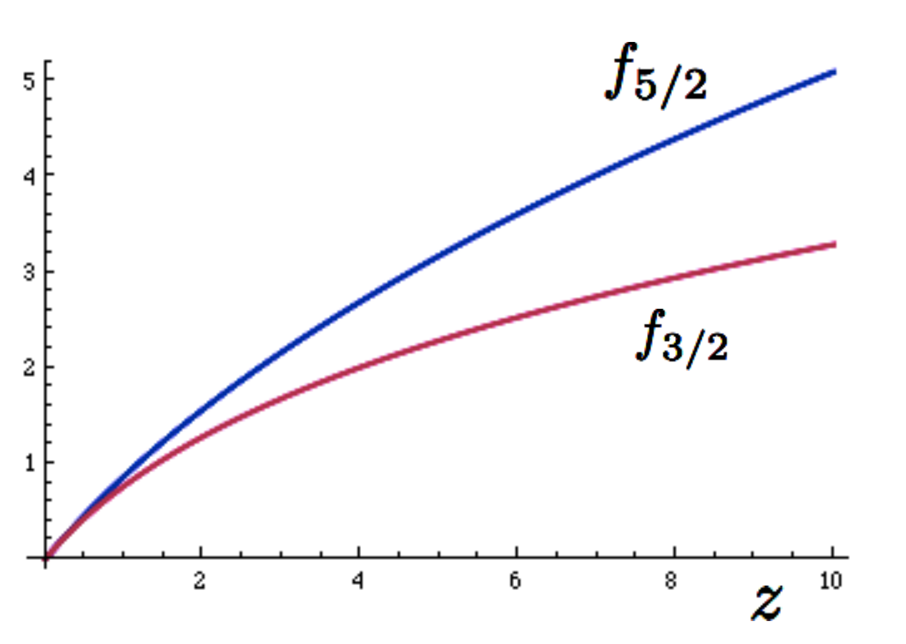
$2s+1$ spin multiplicity

$$\lambda = \frac{h}{\sqrt{2\pi m k_B T}}$$

thermal wave length

$$f_{5/2}(z) = -\sum_{l=1}^{\infty} (-1)^l \frac{z^l}{l^{5/2}}$$

$$f_{3/2}(z) = -\sum_{l=1}^{\infty} (-1)^l \frac{z^l}{l^{3/2}}$$



high-temperature / low-density limit $1 \gg \frac{\lambda^3}{v} \sim z = e^{\beta\mu}$ ➡ $\mu < 0$

fix particle density $$\frac{\lambda^3}{v} = (2s+1)\left\{z - \frac{z^2}{2^{3/2}} + \cdots\right\}$$

➡ $$z = \frac{1}{2s+1}\frac{\lambda^3}{v} + \frac{1}{2^{3/2}(2s+1)^2}\left(\frac{\lambda^3}{v}\right)^2 + \cdots$$

insert

pressure

$$\frac{p}{k_B T} \approx \frac{2s+1}{\lambda^3}\left\{z - \frac{z^2}{2^{5/2}}\right\} \approx \frac{N}{V}\left\{1 + \frac{N}{2^{5/2}(2s+1)}\frac{\lambda^3}{V}\right\}$$

increased pressure

classical ideal gas *quantum correction*

compressibility

$$\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_{T,N} = \frac{V}{Nk_B T}\frac{1}{1 + \frac{\lambda^3 N}{2^{3/2}(2s+1)V}}$$

reduced because Fermions avoid each other

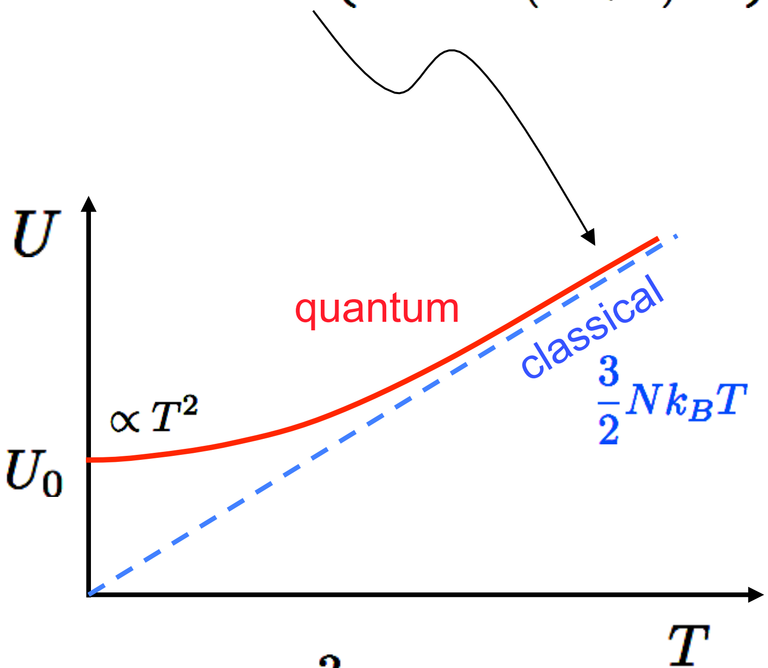
high-temperature / low-density limit

internal energy

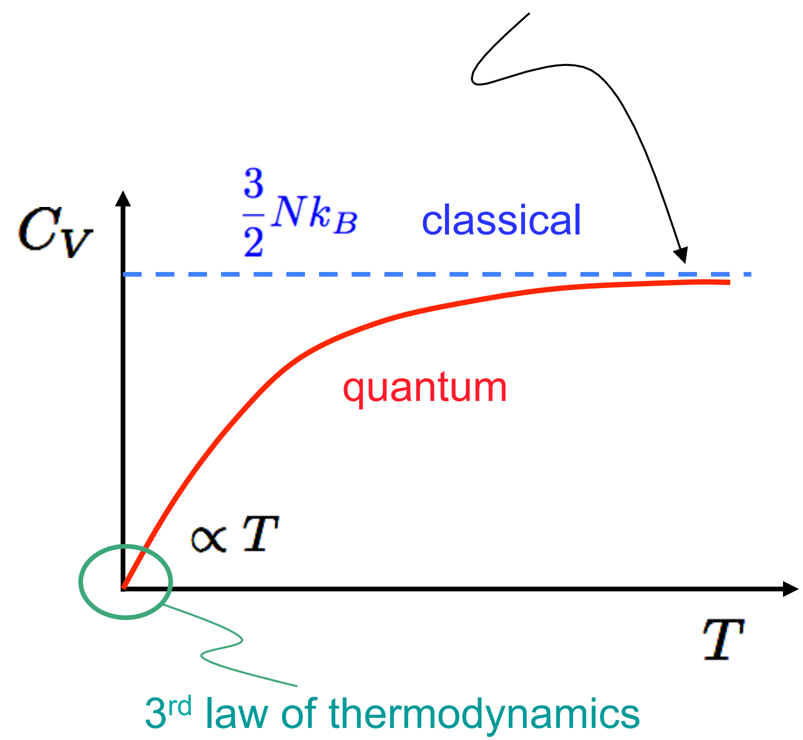
$$U \approx \frac{3}{2} N k_B T \left\{ 1 + \frac{N}{2^{5/2}(2s+1)} \frac{\lambda^3}{V} \right\}$$

heat capacity

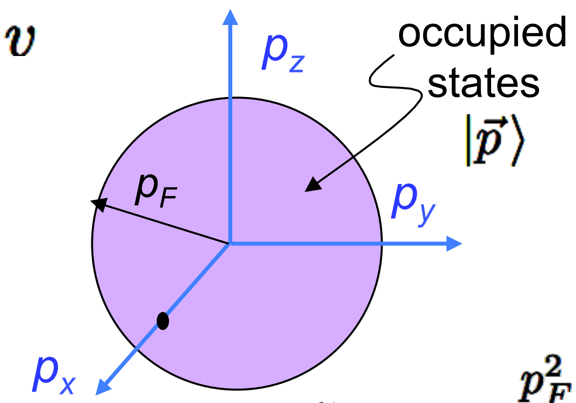
$$C_V = \left(\frac{\partial U}{\partial T} \right)_{V,N} = \frac{3}{2} N k_B \left\{ 1 - \frac{N}{2^{7/2}(2s+1)} \frac{\lambda^3}{V} \right\}$$



note: $U = \frac{3}{2} pV$



low-temperature / high-density limit $1 \ll \frac{\lambda^3}{v}$



$\mu = \epsilon_F = \frac{p_F^2}{2m}$

Fermi energy

T=0: ground state

$$\langle n_{\vec{p}} \rangle = \Theta(\mu - \epsilon_{\vec{p}}) = \begin{cases} 1 \, , & \epsilon_{\vec{p}} < \mu \, , \\ 0 \, , & \epsilon_{\vec{p}} > \mu \, . \end{cases}$$

particle density:

$$\frac{1}{v} = \frac{N}{V} = \frac{2s+1}{h^3} \int d^3p \langle n_{\vec{p}} \rangle = \frac{2s+1}{h^3} \frac{4\pi}{3} p_F^3 \quad \Rightarrow \quad p_F = \left\{ \frac{6\pi^2 \hbar^3}{v(2s+1)} \right\}^{1/3}$$

Fermi momentum

internal (ground state) energy:

$$U_0 = \frac{2s+1}{h^3} V \int d^3p \, \epsilon_{\vec{p}} \langle n_{\vec{p}} \rangle = \frac{3}{5} N \epsilon_F$$

zero-piont pressure:

$$\Rightarrow \quad p_0 = \frac{2}{3} \frac{U}{V} = \frac{2}{5} \frac{N}{V} \epsilon_F$$

low-temperature / high-density limit $1 \ll \frac{\lambda^3}{v}$

T > 0: $T \ll T_F = \epsilon_F / k_B \longrightarrow z = e^{\beta\mu} \gg 1$

approximation

$$f_{5/2}(z) \approx \frac{8}{15\sqrt{\pi}}(\ln z)^{5/2}\left[1 + \frac{5\pi^2}{8(\ln z)^2} + \cdots\right]$$

$$f_{3/2}(z) = \frac{4}{3\sqrt{\pi}}(\ln z)^{3/2}\left[1 + \frac{\pi^2}{8(\ln z)^2} + \cdots\right]$$

$$\ln z = \beta\mu \approx \frac{T_F}{T} \gg q$$

particle density:

$$\frac{1}{v} = \frac{(2s+1)p_F^3}{6\pi^2\hbar^3} = \frac{(2s+1)(2m\epsilon_F)^{3/2}}{6\pi^2\hbar^3} = \frac{2s+1}{\lambda^3} f_{3/2}(z)$$

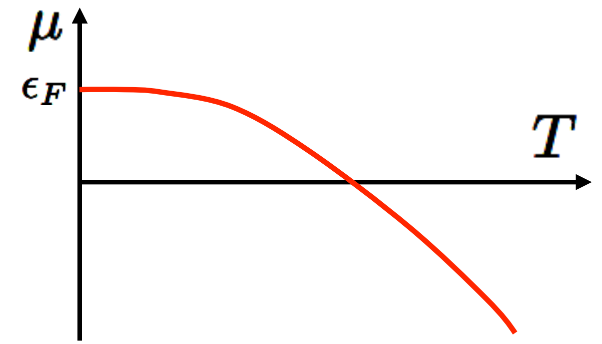
$$(\beta\epsilon_F)^{3/2} = (\beta\mu)^{3/2} + \frac{\pi^2}{8}(\beta\mu)^{-1/2} + \cdots$$

$$\mu(T) = \epsilon_F \left(1 - \frac{\pi^2}{12}\left(\frac{T}{T_F}\right)^2 + \cdots\right)$$

low-temperature / high-density limit $1 \ll \frac{\lambda^3}{v}$

T > 0: $T \ll T_F = \epsilon_F / k_B \rightarrow z = e^{\beta\mu} \gg 1$

$$\mu(T) = \epsilon_F \left(1 - \frac{\pi^2}{12}\left(\frac{T}{T_F}\right)^2 + \cdots\right)$$

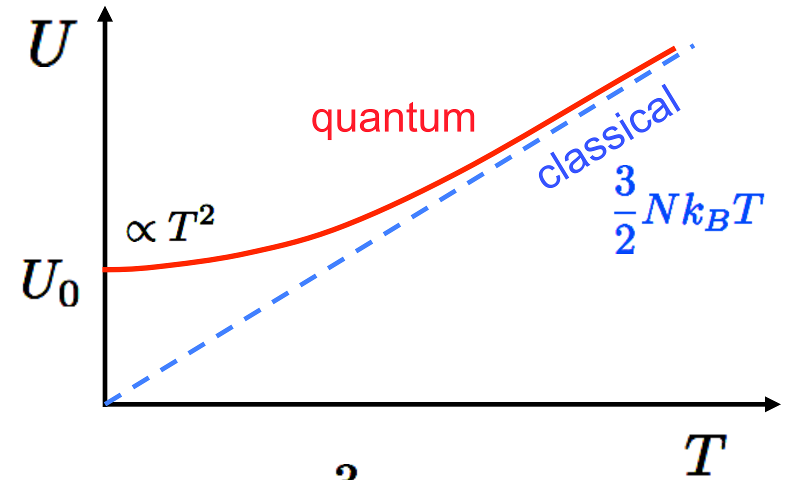


pressure
internal energy:

$$\beta p = \frac{2s+1}{\lambda^3} f_{5/2}(z)$$

$$p = p_0 \left(1 + \frac{5\pi^2}{12}\left(\frac{T}{T_F}\right)^2 + \cdots\right)$$

$$U = U_0 \left(1 + \frac{5\pi^2}{12}\left(\frac{T}{T_F}\right)^2 + \cdots\right)$$



note: $U = \frac{3}{2} pV$