Self-consistent field approximation

partition function from Gaussian transformation

$$Z = C \int_{-\infty}^{+\infty} \left(\prod_i d\phi_i \right) e^{-\beta S(\phi_i, H_i)} \qquad C = \frac{1}{(2\pi k_B T)^{N/2} \sqrt{det J}}$$

$$S(\phi_i, H_i = 0) = -\frac{1}{2} \sum_{i,j} (J^{-1})_{ij} \phi_i \phi_j - \frac{1}{\beta} \sum_i \ln[2\cosh(\beta s \phi_i)]$$

expansion around $\langle \phi_i \rangle = 0 \qquad T > T_c$

$$S(\phi) \approx S_0 + \frac{1}{2} \sum_{i,j} \left\{ (J^{-1})_{ij} \phi_i \phi_j + k_B T \left\langle \frac{\partial^2 \ln[2\cosh(\beta s \phi_i)]}{\partial \phi_i \partial \phi_j} \right\rangle \phi_i \phi_j \right\}$$

Gaussian approximation

Self-consistent field approximation

Gaussian approximation

$$S(\phi) \approx S_0 + \frac{1}{2}\sum_{i,j}\left\{(J^{-1})_{ij} + \beta s^2 \delta_{ij}\left\langle \frac{1}{\cosh^2(\beta s \phi_i)}\right\rangle\right\}\phi_i\phi_j$$

$$\approx S_0 + \frac{1}{2}\sum_{i,j}\{(J^{-1})_{ij} + \beta s^2 \delta_{ij}(1 - \beta^2 s^2 \langle\phi^2\rangle\}\phi_i\phi_j \ .$$

uniform

$$\langle\phi_i^2\rangle = \langle\phi^2\rangle$$

Fourier transform:

$$\phi_i = \frac{1}{N}\sum_{\vec{q}}\phi_{\vec{q}}\, e^{i\vec{q}\cdot\vec{r}_i}$$

$$S(\phi) = S_0 + \frac{1}{2N}\sum_{\vec{q}}\left\{\frac{1}{J(\vec{q})} + \beta s^2(1 - \beta^2 s^2\langle\phi^2\rangle)\right\}\phi_{\vec{q}}\phi_{-\vec{q}}$$

$$= S_0 + \frac{a^d}{2J^2z^2N}\sum_{\vec{q}}\{\kappa q^2 + A + 3Bs^6\beta^3\langle\phi^2\rangle\}\phi_{\vec{q}}\phi_{-\vec{q}}$$

Self-consistent field approximation

$$\phi_{\vec{q}} = Jzm_{\vec{q}}$$

$$\begin{aligned} S &= S_0 + \frac{a^d}{2N}\sum_{\vec{q}} \left\{\kappa q^2 + A + 3B\langle m^2\rangle\right\} m_{\vec{q}} m_{-\vec{q}} \\ &= S_0 + \frac{1}{2}\sum_{\vec{q}} G^{-1}(\vec{q}) m_{\vec{q}} m_{-\vec{q}} \end{aligned}$$

$$\begin{aligned} Z' &= Z_0 {\prod_{\vec{q}}}' \int dm_{\vec{q}}\, dm_{-\vec{q}}\ \exp\left\{-\beta G^{-1}(\vec{q}) m_{\vec{q}}\, m_{-\vec{q}}/2\right\} \\ &= Z_0 {\prod_{\vec{q}}}' \int dm'_{\vec{q}}\, dm''_{\vec{q}}\ \exp\left\{-\beta G^{-1}(\vec{q}) ({m'_{\vec{q}}}^2 + {m''_{\vec{q}}}^2)/2\right\} \end{aligned}$$

${\prod_{\vec{q}}}'$ runs only over a half-space of $\vec{q}$ e.g. $\{\vec{q} \mid q_z \geq 0\}$

self-consistent field

$$\langle m^2 \rangle = \frac{1}{L^d}\int d^d r \, \langle m(\vec{r})^2 \rangle = \frac{1}{N^2}\sum_{\vec{q}} \langle m_{\vec{q}} m_{-\vec{q}} \rangle$$

$$= \frac{Z_0}{Z' L^d} {\prod_{\vec{q}}}' \int dm_{\vec{q}}\, dm_{-\vec{q}}\, m_{\vec{q}}\, m_{-\vec{q}}\, \exp\left\{-\beta G^{-1}(\vec{q}) m_{\vec{q}}\, m_{-\vec{q}}/2\right\}$$

$$\langle m^2 \rangle = \frac{k_B T}{L^d}\sum_{\vec{q}} G(\vec{q}) = \frac{1}{L^d}\sum_{\vec{q}} \frac{k_B T}{A + 3B\langle m^2 \rangle + \kappa q^2}$$

self-consistence equation for fluctuations $\langle m^2 \rangle$

Self-consistent field approximation - susceptibility

$$\langle m^2 \rangle = \frac{k_B T}{L^d} \sum_{\vec{q}} G(\vec{q}) = \frac{1}{L^d} \sum_{\vec{q}} \frac{k_B T}{A + 3B\langle m^2 \rangle + \kappa q^2}$$

susceptibility

$$\chi(T) \quad = \beta \frac{1}{L^d} \int d^d r \; d^d r' \left\{ \langle m(\vec{r}) m(\vec{r}\,') \rangle - \langle m(\vec{r}) \rangle \langle m(\vec{r}\,') \rangle \right\}$$

$$= \beta \langle m^2_{\vec{q}=0} \rangle = G(\vec{q} = 0) = \frac{1}{A + 3B\langle m^2 \rangle} \qquad T > T_c$$

$$\Rightarrow \quad \chi^{-1} = [A + 3B\langle m^2 \rangle] = A + \frac{3B k_B T}{L^d} \sum_{\vec{q}} \frac{1}{\chi^{-1} + \kappa q^2}$$

Self-consistent field approximation - instabillity

$$\chi^{-1} = [A + 3B\langle m^2\rangle] = A + \frac{3Bk_BT}{L^d}\sum_{\vec{q}}\frac{1}{\chi^{-1} + \kappa q^2}$$

instability at $T \to T_c^*$ $\quad \chi \to \infty \longrightarrow \chi^{-1} \to 0$

$$A = -\frac{3Bk_BT}{L^d}\sum_{\vec{q}}\frac{1}{\kappa q^2} \longrightarrow \frac{1}{L^d}\sum_{\vec{q}} \to \int \frac{d^dq}{(2\pi)^d} \to \frac{C_d}{(2\pi)^d}\int_0^{\Lambda} dq \frac{q^{d-1}}{\kappa q^2}$$

cutoff: $\Lambda \sim a^{-1}$

$$A_c = Jza^{-d}\left(\frac{T_c^*}{T_c} - 1\right)$$

$$= -\frac{Jza^{-d}}{s^2}\frac{C_dk_BT_c^*}{(2\pi)^d}\int_0^{\Lambda} dq\, \frac{q^{d-1}}{\kappa q^2}$$

$$T_c^* = \frac{T_c}{1 + \frac{C_dz}{(2\pi)^d}\frac{(\Lambda a)^{d-2}}{d-2}}$$

Self-consistent field approximation - critical exponents

$$\chi^{-1} = A + \frac{3Bk_BTC_d}{(2\pi)^d}\int_0^\Lambda dq \frac{q^{d-1}}{\chi^{-1}+\kappa q^2} \quad \Longleftrightarrow \quad 0 = A_c + \frac{3Bk_BT_c^*C_d}{(2\pi)^d}\int_0^\Lambda dq \frac{q^{d-1}}{\kappa q^2}$$

temperature dependence of $\chi(T)$ for $T \to T^*_{c+}$

$$\begin{aligned}
\chi^{-1} &= (A - A_c) + \frac{3BC_d}{(2\pi)^d}\int_0^\Lambda dq\, q^{d-1}\left[\frac{k_BT}{\chi^{-1}+\kappa q^2} - \frac{k_BT_c^*}{\kappa q^2}\right] \\
&\approx (A - A_c) - \frac{3BC_dk_BT_c^*}{(2\pi)^d\kappa}\int_0^\Lambda dq \frac{q^{d-3}}{1+\chi\kappa q^2} \qquad \Leftarrow \quad x = q\xi \\
&= (A - A_c) - \frac{3BC_dk_BT_c^*}{(2\pi)^d\kappa}\{\kappa\chi\}^{(2-d)/2}\underbrace{\int_0^{\Lambda\sqrt{\kappa\chi}} dx \frac{x^{d-3}}{1+x^2}}
\end{aligned}$$

correction length $\xi^2 = \kappa\chi$

cutoff necessary for $d \geq d_c = 4$

Self-consistent field approximation - critical exponents

$d \geq d_c = 4$

$$\chi^{-1} \approx (A - A_c) - \frac{3BC_d k_B T_c^*}{(2\pi)^d \kappa} \{\kappa\chi\}^{(2-d)/2} \frac{\{\Lambda(\kappa\chi)^{1/2}\}^{d-4}}{d-4}$$

$$= \frac{k_B}{a^d s^2} \underbrace{(T - T_c^*)}_{\propto\, \tau} - \frac{C_d z^2}{2(2\pi)^d} \frac{T_c^*}{T_c} \frac{(\Lambda a)^{d-4}}{d-4} \chi^{-1}$$

$$\Rightarrow \quad \chi(T) = \frac{a^d s^2}{k_B (T - T_c^*)} \left\{ 1 + \frac{C_d z^2}{2(2\pi)^d} \frac{T_c^*}{T_c} \frac{(\Lambda a)^{d-4}}{d-4} \right\}^{-1} \propto |\tau|^{-1}$$

$\chi \propto |\tau|^{-\gamma}$

$\xi \propto |\tau|^{-\nu}$

$\Rightarrow$

$\xi^2 = \kappa\chi$

Fisher scaling

$\gamma = (2 - \eta)\nu$

$\Rightarrow$

$\gamma = 1$ *mean field exponents*

$\nu = 1/2$ but

$\eta = 0$ $T_c^* < T_{\mathrm{cmf}}$

Self-consistent field approximation - critical exponents

$d < d_c = 4$

$$\chi^{-1} = \underbrace{(A - A_c)}_{\propto\, \tau} - \frac{3BC_d k_B T_c^*}{(2\pi)^d \kappa^{d/2}} K_d \chi^{(2-d)/2}$$

$$\frac{d-2}{2} < 1$$

regime 1: $\chi^{(2-d)/2}$ dominant T "close" to T_c^*

$$\chi \propto |\tau|^{-\gamma} \qquad \gamma = \frac{2}{d-2}$$

regime 2: χ^{-1} dominant T "far" from T_c^*

$$\chi \propto |\tau|^{-\gamma} \qquad \gamma = 1$$

mean field