6.1. Lorentz covariance and critical dimension of bosonic string theory

The quantum Lorentz generator M^{-I} is given by the expression

$$M^{-I} = x_0^- p^I - \frac{1}{4\alpha' p^+}\Big[x_0^I\big(L_0^\perp + a\big) + \big(L_0^\perp + a\big)x_0^I\Big] - \frac{i}{\sqrt{2\alpha'}p^+}\sum_{n=1}^{\infty}\frac{1}{n}\big(L_{-n}^\perp\alpha_n^I - \alpha_{-n}^I L_n^\perp\big). \tag{6.1}$$

Recall that the definition of the transverse Virasoro generators is

$$L_n^\perp = \frac{1}{2}\sum_{p\in\mathbb{Z}}\alpha_{n-p}^I\alpha_p^I \quad (\text{for } n\neq 0)\,, \qquad L_0^\perp = \frac{1}{2}\alpha_0^I\alpha_0^I + \sum_{p=1}^{\infty}\alpha_p^I\alpha_p^I\,, \tag{6.2}$$

that $p^I = \sqrt{2\alpha'}\alpha_0^I$ and also that

$$[\alpha_m^I, \alpha_n^J] = m\,\eta^{IJ}\delta_{m+n,0}\,, \qquad [x_0^I, p^J] = i\,\eta^{IJ}\,. \tag{6.3}$$

Here and throughout this exercise two repeated upper Lorentz indices are contracted with the transverse euclidean metric $\eta^{IJ} = \delta^{IJ}$.

Show that the following equality holds (for $m > 0$):

$$\langle p^+, \vec{0}\,|\,\alpha_m^I\,[M^{-I}, M^{-J}]\,\alpha_{-m}^J\,|p^+, \vec{0}\rangle = -\frac{m^2}{\alpha'(p^+)^2}\left[m\left(1 - \frac{D-2}{24}\right) + \frac{1}{m}\left(\frac{D-2}{24} + a\right)\right] \tag{6.4}$$

and hence compute the critical dimension and normal-ordering shift for $L_0^\perp$.

Hint: recall (or show) that

$$[L_m^\perp, \alpha_n^J] = -n\alpha_{n+m}^J\,, \qquad [L_m^\perp, x_0^J] - i\sqrt{2\alpha'}\alpha_m^J\,, \tag{6.5}$$

then carefully commute the M's through the α's. Recall also the action of M^{-I} on the external states and the action of $L_0^\perp$!

6.2. State counting

The Fock space $\mathcal{H}$ of the open string is generated from the ground state $|p^+, \vec{p}_T\rangle$ by the action of the creation operators α_{-n}^I, $n > 0$. Recall that the ground state is annihilated by the positive-level oscillators.

We want to derive the *generating function* for the bosonic string spectrum, that is a function $f(q) = \sum_{n=0}^{\infty} c(n)q^n$ such that the coefficient $c(n)$ counts the number of states at level n.

Argue that is given by (q times) the trace

$$\mathrm{tr}_{\mathcal{H}}\left(q^{\alpha' M^2}\right)\,, \tag{6.6}$$

where

$$M^2 = \frac{1}{\alpha'}\left(-1 + \sum_{n=1}^{\infty} n\,(a_n^I)^\dagger a_n^I\right) \tag{6.7}$$

is the mass-squared operator.
Moreover, compute the trace and show that

$$\mathrm{tr}_{\mathcal{H}}\left(q^{\alpha' M^2}\right) = \frac{1}{\eta^{24}(q)}\,, \tag{6.8}$$

where

$$\eta(q) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \tag{6.9}$$

is the Dedekind eta function.