## 8.1. Commutation relations for the closed string oscillators

The canonical commutation relations of the dynamical variables $X^I$ and their conjugate momenta $P^{\tau J}$ are

$$[X^I(\tau,\sigma)\,,\,P^{\tau J}(\tau,\sigma')] = i\,\eta^{IJ}\,\delta(\sigma-\sigma')\,. \tag{8.1}$$

**a)** Show that the above canonical commutation relations imply the (distributional) equality

$$\Big[(\dot{X}^I \pm X'^I)(\tau,\sigma)\,,(\dot{X}^J \pm X'^J)(\tau,\sigma')\Big] = \pm 4\pi\alpha' i\,\frac{d}{d\sigma}\delta(\sigma-\sigma')\,. \tag{8.2}$$

**b)** Consider eq. (8.2) with the minus sign. Recalling the mode expansion

$$X^\mu(\tau,\sigma) = x_0^\mu + \sqrt{2\alpha'}\alpha_0^\mu\tau + i\frac{\alpha'}{\sqrt{2}}\sum_{n\neq 0}\frac{e^{-in\tau}}{n}(\alpha_n^\mu e^{in\sigma} + \tilde{\alpha}_n^\mu e^{-in\sigma})\,, \tag{8.3}$$

show that it follows from the canonical commutation relations eq. (8.2) that

$$[\alpha_m^I, \alpha_n^J] = m\eta^{IJ}\delta_{m+n,0}\,. \tag{8.4}$$

*Hint:* exploit the orthogonality of the functions $e^{iq\sigma}$ with $q \in \mathbb{Z}$.

**c)** Show that

$$[x_0^I + \sqrt{2\alpha'}\alpha_0^I\tau,\ \dot{X}^J(\tau,\sigma')] = i\alpha'\eta^{IJ}\,. \tag{8.5}$$

Use this result to show that

$$[x_0^I, \alpha_n^J] = 0 \ \text{ for } n \neq 0\,, \quad [x_0^I, \alpha_0^J] = i\sqrt{\frac{\alpha'}{2}}\eta^{IJ}\,, \qquad [\alpha_0^I, \alpha_0^J] = 0\,. \tag{8.6}$$

*Hint:* the set of functions $e^{in\sigma}$ with $n \in \mathbb{Z}$ is complete on the interval $\sigma \in [0, 2\pi]$.

## 8.2. Action of $L_0^\perp - \bar{L}_0^\perp$

We define $L_0^\perp - \bar{L}_0^\perp := P$.

**a)** Show that the equation

$$\Big[P, X^I(\tau,\sigma)\Big] = i\frac{\partial X^I}{\partial\sigma} \tag{8.7}$$

implies that the following equality holds for any finite $\sigma_0$.

$$X^I(\tau,\sigma+\sigma_0) = e^{-iP\sigma_0}X^I(\tau,\sigma)e^{iP\sigma_0}\,. \tag{8.8}$$

**b)** Show that

$$e^{-iP\sigma_0}\,\big[\dot{X}^I + X'^I\big](\tau,\sigma)\,e^{iP\sigma_0} \;=\; \big[\dot{X}^I + X'^I\big](\tau,\sigma+\sigma_0)\,. \tag{8.9}$$

**c)** Using eq. (8.9), compute the action of $\sigma$-translations on the oscillators $\alpha_n^I$, $\bar{\alpha}_n^I$, i.e. compute $e^{-iP\sigma_0}\alpha_n^I e^{iP\sigma_0}$ and $e^{-iP\sigma_0}\bar{\alpha}_n^I e^{iP\sigma_0}$.

**d)** Consider the state

$$|U\rangle = \alpha^I_{-m}\bar{\alpha}^J_{-n}\,|p^+,\vec{p}_T\rangle\,, \qquad m,n>0\,. \tag{8.10}$$

Compute the $\sigma$-translated state $e^{-iP\sigma_0}|U\rangle$. What is the condition that makes this state invariant?

## 8.3. Maxwell and Kalb-Ramond fields

Light-cone gauge is a valid gauge choice in many theories, not only string theory. We will study a gauge field theory where the light-cone gauge choice allows us to extract most easily physical information.

The Kalb-Ramond field $B_{\mu\nu}$ is an antisymmetric Lorentz tensor with the gauge symmetry transformation

$$\delta B_{\mu\nu} = \partial_\mu\epsilon_\nu - \partial_\nu\epsilon_\mu\,. \tag{8.11}$$

We define the field strength $H_{\mu\nu\rho}$ and the action $S_{\mathrm{KR}}$ for $B_{\mu\nu}$ as

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}\,, \quad \text{and} \qquad S_{\mathrm{KR}} = -\frac{1}{12}\int \mathrm{d}^D x\, H_{\mu\nu\rho}H^{\mu\nu\rho}\,. \tag{8.12}$$

**a)** Show that the action of eq. (8.12) is invariant under the gauge transformations of eq. (8.11). Derive the equations of motion and express them in momentum space.

**b)** Show that the gauge transformation of $B_{\mu\nu}$ has a redundancy

$$\epsilon'_\mu = \epsilon_\mu + \partial_\mu\lambda\,, \tag{8.13}$$

under which $\delta B_{\mu\nu}$ is invariant. Express the gauge transformations in light-cone momentum space and show that it is possible to gauge away the component $\epsilon_+$, so that the effective gauge transformation of $B_{\mu\nu}$ is generated by $\epsilon_-$ and $\epsilon_i$.

**c)** We now want to enforce light-cone gauge. Express the gauge transformations in momentum space. Show that, by a sensible choice of $\tilde{\epsilon}(p)$, you can gauge away all the $+$-components of the light-cone gauge field $(B^{+-}, B^{+I}, B^{-I}, B^{IJ})$; show that the equations of motion in momentum space are drastically simplified in this gauge. Count the total number of *independent* degrees of freedom of the gauged Kalb-Ramond field.

**d)** In four dimensions, we can define a dual field $\tilde{H}_\mu$ as

$$\tilde{H}_\mu = \epsilon_{\mu\nu\rho\sigma}H^{\nu\rho\sigma}\,. \tag{8.14}$$

Using the results found in the previous parts of this problem, show that this dual field can be expressed as the derivative of a single scalar field. What does this imply for the Kalb–Ramond field in four dimensions?