10.1. Dualities in the toroidal compactification with Kalb-Ramond background

In sheet 9 you computed the spectrum of a compactification on a square torus T^2 with radius R and nonvanishing background Kalb-Ramond field $B_{23} = b/(2\pi\alpha')$. We want now to understand the duality symmetries in the spectrum. Recall that the spectrum is described by the equations

$$M^2 = (\tfrac{n_2}{R} + \tfrac{m_3 R}{\alpha'} b)^2 + (\tfrac{n_3}{R} - \tfrac{m_2 R}{\alpha'} b)^2 + (\tfrac{m_3 R}{\alpha'})^2 + (\tfrac{m_2 R}{\alpha'})^2 + \tfrac{2}{\alpha'}(N^\perp + \bar{N}^\perp - 2)\,, \quad (10.1)$$

$$N^\perp - \bar{N}^\perp = n_2 m_2 + n_3 m_3\,. \quad (10.2)$$

a) Consider the transformation consisting of the shift in the background field given by

$$b \mapsto b' = b + \frac{\alpha'}{R^2}\,\ell\,, \qquad \ell \in \mathbb{Z} \quad (10.3)$$

and leaving R unchanged. Find the compensating change in the quantum numbers, analogous to the exchange $n \leftrightarrow m$ in the ordinary case, that leaves the spectrum invariant. *Hint:* $n_2 \to n_2 - \ell m_3$ is one of the needed changes.

b) Consider the transformation

$$b \mapsto b' = -b\,, \qquad R \mapsto R' = \frac{\alpha'}{R}\frac{1}{\sqrt{1+b^2}}\,. \quad (10.4)$$

Notice that for $b = 0$ this is the ordinary T-duality transformation. Find the compensating change in momentum and winding modes that leaves the spectrum invariant.

c) Consider the transformation

$$b \mapsto b' = -b\,, \qquad R \mapsto R' = R\,. \quad (10.5)$$

Find the compensating change in momentum and winding modes that leaves the spectrum invariant. *Hint:* think carefully about eq. (10.2).

10.2. Yang–Mills three gluon vertex

In this exercise we look at a generalisation of QED, where the gauge field A_μ takes values in the Lie algebra $\mathfrak{su}(2)$ of SU(2). This is the simplest example of a *Yang–Mills theory.* In Yang–Mills, the gauge boson is called *gluon.*

Recall that $\mathfrak{su}(2)$ is the Lie algebra spanned by the Pauli matrices $\tau^a = \frac{1}{\sqrt{2}}\sigma^a$, $a = 1, 2, 3$, satisfying

$$[\tau^a, \tau^b] = i\epsilon^{abc}\tau^c\,. \quad (10.6)$$

Here, ϵ^{abc} is the totally antisymmetric Levi-Civita symbol; moreover, repeated indices a, b, c are to be intended as summed over[3]. Here, the ϵ^{abc} are called the *structure constants* of the algebra $\mathfrak{su}(2)$.

[3] This is a slight abuse, since, for the usual notion of "colour" employed f.e. in QCD, a, b, c, represent – roughly speaking – a color-anticolor pair of indices.

A $\mathfrak{su}(2)$-valued gauge field A_μ is then of the form $A_\mu(x) := A^a_\mu(x)\,\tau^a$, where the components $A^a_\mu(x)$ are ordinary (real-valued) functions. The Yang–Mills Lagrangian density is

$$\mathcal{L}_{\rm YM} = -\frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu}\,, \tag{10.7}$$

the field strength $F^a_{\mu\nu}$ is defined as

$$F^a_{\mu\nu} := \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\,\epsilon^{abc} A^b_\mu A^c_\nu\,, \tag{10.8}$$

where g is the dimensionless coupling constant.

a) Show that the infinitesimal gauge transformation of the gauge field

$$\delta A^a_\mu \;=\; \partial_\mu \lambda^a + g\,\epsilon^{abc} A^b_\mu \lambda^c \tag{10.9}$$

induces the variation

$$\delta F^a_{\mu\nu} \;=\; g\epsilon^{abc}\lambda^c F^b_{\mu\nu}\,. \tag{10.10}$$

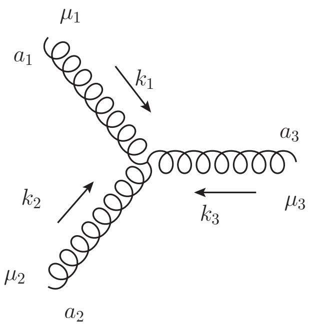
Here $\lambda^a(x)$ are local infinitesimal functions. Show then that $\delta\mathcal{L}_{\rm YM} = 0$.

b) Start from the Lagrangian density eq. (10.7) and expand it in a sum of quadratic, cubic and quartic terms.

We want to derive heuristically the form of the three-gluon vertex. Consider the connected component of the correlator

$$M'^{a_1,a_2,a_3}_{\mu_1\mu_2\mu_3} = \langle 0|\mathcal{T}\Big\{A^{a_1}_{\mu_1}(x_1)A^{a_2}_{\mu_2}(x_2)A^{a_3}_{\mu_3}(x_3)\int d^4y\,\mathcal{L}_{A^3}(y)\Big\}|0\rangle\,, \tag{10.11}$$

assuming the theory is properly quantized. Show that in momentum space the result reads



$$\tilde{M}^{a_1a_2a_3}_{\mu_1\mu_2\mu_3} = -\,g\,\epsilon^{a_1a_2a_3}\Big[(k_2-k_3)_{\mu_1}\eta_{\mu_2\mu_3} + (k_3-k_1)_{\mu_2}\eta_{\mu_3\mu_1} + (k_1-k_2)_{\mu_3}\eta_{\mu_1\mu_2}\Big]$$

Hint: use Wick's theorem! In order to compute the connected component, do not contract fields sitting at the same point in spacetime. Recall that the contraction of two fields yields the propagator, which in the case of a gluon is

$$\langle 0|\mathcal{T} A^a_\mu(x) A^b_\nu(y)|0\rangle = \varDelta^{ab}_{\mu\nu}(x-y)\,. \tag{10.12}$$

Go then to momentum space; "amputate" the Green's function by just substituting $\tilde{\varDelta}^{ab}_{\mu\nu}(p) \to \delta^{ab}\eta_{\mu\nu}$ (recall that the LSZ reduction formula relates amputated Green's functions with S-matrix elements).

c) Compute the amplitude

$$A^{a_1a_2a_3}(k_1,k_2,k_3) \;=\; \epsilon^{\mu_1}_{\lambda_1}\epsilon^{\mu_2}_{\lambda_2}\epsilon^{\mu_3}_{\lambda_3}\,\tilde{M}^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}\,. \tag{10.13}$$