13.1. Jacobi theta functions

An important set of functions in the theory of modular forms[1] are the so-called Jacobi theta functions. Here we consider the three functions

$$\begin{aligned}
\theta_2(\tau) &= \sum_{n\in\mathbb{Z}+1/2} q^{\frac{n^2}{2}} &&= 2q^{\frac{1}{8}}\prod_{n=1}^{\infty}(1-q^n)(1+q^n)^2\,,\\
\theta_3(\tau) &= \sum_{n\in\mathbb{Z}} q^{\frac{n^2}{2}} &&= \prod_{n=1}^{\infty}(1-q^n)(1+q^{n-\frac{1}{2}})^2\,,\\
\theta_4(\tau) &= \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{n^2}{2}} &&= \prod_{n=1}^{\infty}(1-q^n)(1-q^{n-\frac{1}{2}})^2\,,
\end{aligned} \tag{13.1}$$

where $q = e^{2\pi i\tau}$. Here and from now on, $\tau \in \mathbb{H}^+$, where $\mathbb{H}^+$ is the complex upper half plane.

a) Show that

$$\begin{aligned}
\theta_2(\tau+1) &= e^{i\frac{\pi}{4}}\,\theta_2(\tau)\,,\\
\theta_3(\tau+1) &= \theta_4(\tau)\,,\\
\theta_4(\tau+1) &= \theta_3(\tau)\,.
\end{aligned} \tag{13.2}$$

b) In order to compute the behaviour of θ's under the inversion $\tau \to -\frac{1}{\tau}$, we can employ the so-called *Poisson resummation formula*. Prove that, for any $f : \mathbb{R} \mapsto \mathbb{C}$ smooth and small at infinity[2], the following equality holds

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \tilde{f}(n)\,, \quad \text{where} \quad \tilde{f}(y) := \int_{-\infty}^{\infty} e^{2\pi i x y} f(x)\, dx\,. \tag{13.3}$$

Hint: show that the function $g(x) := \sum_{n\in\mathbb{Z}} f(x+n)$ is well-defined and periodic; exploit this fact and that $\sum_n f(n) \equiv g(0)$.

c) Using the Poisson resummation formula, show that

$$\begin{aligned}
\theta_2(-\tfrac{1}{\tau}) &= \sqrt{-i\tau}\,\theta_4(\tau)\,,\\
\theta_3(-\tfrac{1}{\tau}) &= \sqrt{-i\tau}\,\theta_3(\tau)\,,\\
\theta_4(-\tfrac{1}{\tau}) &= \sqrt{-i\tau}\,\theta_2(\tau)\,.
\end{aligned} \tag{13.4}$$

d) The Dedekind η function is defined as

$$\eta(\tau) := q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)\,. \tag{13.5}$$

[1] Modular forms are holomorphic functions $f : \mathbb{H}^+ \mapsto \mathbb{C}$ that, loosely speaking, "transform nicely" under (finite index subgroups of) $\mathrm{SL}(2,\mathbb{Z})$. Here $\mathrm{SL}(2,\mathbb{Z})$ acts on $z \in \mathbb{H}^+$ as $\mathrm{SL}(2,\mathbb{Z}) \ni \gamma = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) : z \mapsto \frac{a\,z+b}{c\,z+d}$.

[2] A sufficient condition is that $f \in L^1(\mathbb{R})$.

Show that

$$[\eta(\tau)]^3 = \frac{1}{2}\theta_2(\tau)\,\theta_3(\tau)\,\theta_4(\tau)\,. \tag{13.6}$$

Hence, show that

$$\eta(\tau+1) = e^{i\frac{\pi}{12}}\,\eta(\tau)\,, \qquad \eta(-\tfrac{1}{\tau}) = \sqrt{-i\tau}\,\eta(\tau)\,. \tag{13.7}$$

13.2. Modular properties of Dedekind η function

In this exercise, we want to derive directly the modular transformation

$$\eta(-\tfrac{1}{\tau}) = \sqrt{-i\tau}\,\eta(\tau) \tag{13.8}$$

of the Dedekind eta function.

Consider first $\tau = iy$, y real and positive. We will establish the transformation in eq. (13.8) along the imaginary axis since we can then analytically continue the result in the whole $\mathbb{H}^+$. From now on, we thus fix $\tau = iy$ and work with positive real y.

a) Show that eq. (13.8) is equivalent to the equality

$$\sum_{m=1}^{\infty}\frac{1}{m}\frac{1}{1-e^{2\pi my}} - \sum_{m=1}^{\infty}\frac{1}{m}\frac{1}{1-e^{2\pi m/y}} - \frac{\pi}{12}\left(y-\frac{1}{y}\right) = -\frac{1}{2}\log y\,. \tag{13.9}$$

b) Our aim is to prove eq. (13.9) via residue calculus.

We now fix $y > 0$ and consider the function

$$F_n(z) = -\frac{1}{8z}\cot\Big[i\pi(n+\tfrac{1}{2})z\Big]\,\cot\left[\frac{\pi(n+\frac{1}{2})z}{y}\right]. \tag{13.10}$$

Let C be the parallelogram in the z complex plane that joins the vertices $y, i, -y, -i$. Compute the integral $\int_C F_n(z)\,dz$ via the residue theorem and show that the limit for $n\to\infty$ of $2\pi i$ times the sum of residues equals the l.h.s. of eq. (13.9). *Hint:* there are $4n$ simple poles and a triple pole you have to consider.

c) What is left to do is to show that

$$\lim_{n\to\infty}\int_C F_n(z)\,dz = -\frac{1}{2}\log y\,. \tag{13.11}$$

The tricky part is to show that we can liberally exchange sum and integration, so that

$$\lim_{n\to\infty}\int_C F_n(z) = \int_C \lim_{n\to\infty} zF_n(z)\frac{dz}{z}\,. \tag{13.12}$$

You can assume this[3], and then show that this integral equals $-\frac{1}{2}\log y$. This completes the proof.

[3]Notice that the sequence $\{zF_n(z)\}$ is uniformly bounded and convergent almost everywhere on C. Moreover, each $z\,F_n$ is (Lebesgue) integrable on each side of C; then eq. (13.12) is a consequence of Lebesgue's dominated convergence theorem.