Exercise 1. ***Velocity addition***

In this exercise we work with the matrix form of the Lorentz transformations. Assume $c = 1$.

1. Consider two successive boosts with velocities v_1 and v_2, respectively, along the $\hat{x}$ direction. Derive the Lorentz transformation associated to the combination of these two boosts, and deduce the velocity addition formula (for the case when the two velocities are parallel). Show that for $v_{1,2} \leq 1$ you cannot exceed the speed of light.
   Recall that in the matrix notation the boost along the $\hat{x}$ axis takes the following form:

$$\Lambda(v) = \begin{pmatrix} \gamma & -\gamma v & & \\ -\gamma v & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \tag{1}$$

   where v is the velocity of the boost, $\gamma = \gamma(v) = \frac{1}{\sqrt{1-v^2}}$, and the elements not shown explicitly are equal to 0.

2. Applying first a Lorentz boost along the $\hat{x}$ axis with velocity v_x, and then one along the $\hat{y}$ axis with velocity v_y, show that the combined transformation can alternatively be written as a composition of a boost and a rotation. What is the magnitude and direction of the velocity of the boost in this case? As before, prove that the resulting speed does not exceed the speed of light.
   The matrix describing a boost along the $\hat{y}$ axis takes the form:

$$\Lambda(v_y) = \begin{pmatrix} \gamma & 0 & -\gamma v_y & \\ 0 & 1 & 0 & \\ -\gamma v_y & 0 & \gamma & \\ & & & 1 \end{pmatrix} \tag{2}$$

Exercise 2. ***Twin paradox***

- In the year 2043 (fraternal) twins Alice and Bob are born on Earth. On 5 April 2063 Alice becomes a pilot for the newly developed Warp Drive spaceship, the Phoenix. With a constant velocity $v = \frac{\sqrt{3}}{2}$ with respect to the Earth she flies to the planet Vulcan (located 16 light years from the Earth), and then returns to Earth with velocity $(-v)$. Assume that all the accelerations were instantaneous.
  Alice thus returns to Earth in the year 2100 and finds Bob to be 57 years old. How old is she (i.e., how many years have passed from her perspective)? What is the explanation of this phenomenon?

- From Alice's point of view, it was Bob who was moving. One could naively repeat the calculations done before and reason that it is Bob who should be younger. Argue that this solution would be incorrect, thus resolving the *Twin Paradox*.
- Now assume that the Universe is a cylinder and Alice can return from Vulcan to Earth without the need of turning back. How can you resolve the paradox in this case?

Exercise 3. ***Relativistic Doppler effect***

Consider a frame $\mathcal{O}'$, moving away from Earth (system $\mathcal{O}$), in which a light source is at rest. In $\mathcal{O}'$, light is emitted at an angle θ' with respect to the x'-axis, and its frequency is ν'.

1. Assuming the Earth as fixed and the system $\mathcal{O}'$ moving with constant velocity v along the x-axis (i.e., x- and x'-axes are parallel, see Figure 1), find the angle θ the light ray makes with the x-axis in the system $\mathcal{O}$ as well as the frequency ν of the light in this system.

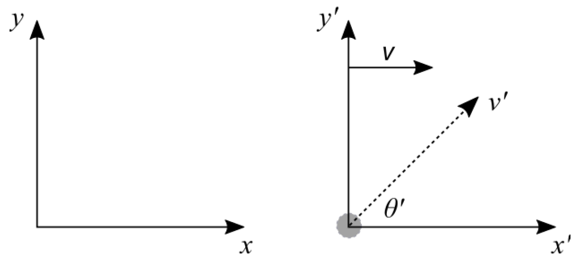

Figure 1: Illustration for the Exercise 3.

2. Assuming $v = 0.7c$, find the angle θ' for which $\nu = \nu'$, that is the Doppler shift vanishes. What do you expect in the non-relativistic case ($v \ll c$)?

Exercise 4. ***How to park your car***

a The fastest car as of 2015 is the Henessy Venom GT, reaching the speed of 270 mph ($\approx 435\frac{km}{h}$). The car's length in its own reference frame is 4,66 m. What is its length in the reference frame of the racing track when the car reaches its maximum speed?

  As of today, the precision of length measurements reaches down to the value of $\frac{\Delta L}{L} \sim 10^{-11}$. Are we able to measure the length contraction of the car?

b Centuries in the future, consider a car achieving a velocity of $v = \frac{\sqrt{3}}{2}$, and measuring 4 meters. The owner of the car goes to visit a friend who possesses a garage measuring 2 meters.
  The friend sees the car approaching with its maximum velocity. What is the length of the car that he observes? Will the car fit into the garage?
  Conversely, what is the length of the garage that the driver of the speeding car perceives? Will the car fit into the garage from his point of view?

  Solve this exercise by drawing a space-time diagram depicting the situation.