Exercise 1. *Manifold S^2, Part 1*

i) Show that the 2-sphere, i.e. the surface

$$S^2 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}\,,$$

is a differentiable manifold. To this end, consider the open cover $O_i^\pm := \{\pm x_i > 0\}$ of S^2, and the maps $\psi_i^\pm : O_i^\pm \to D$, which project the open sets $O_i^\pm$ to the open disc $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$, respectively. Show that the transition functions are smooth. [Hint: It is sufficient to show this for just one transition function.]

ii) For this part use only the chart $\psi_1^+ : (x_1, x_2, x_3) \mapsto (x_2, x_3)$. Find the components a^μ and b^μ of the two basis vectors

$$X_u = \frac{\partial}{\partial u} = a^\mu \frac{\partial}{\partial x^\mu}\,, \qquad X_v = \frac{\partial}{\partial v} = b^\mu \frac{\partial}{\partial x^\mu}\,, \qquad \mu = 1, 2, 3$$

w.r.t. the partial derivatives of $\mathbb{R}^3$ by calculating $X_u(f|_{S^2})$ and $X_v(f|_{S^2})$ at a point $p \in S^2$, where f is a differentiable function on $\mathbb{R}^3$, i.e., calculate

$$X_u(f|_{S^2}) = \frac{\partial}{\partial u}(f \circ (\psi_1^+)^{-1})|_{\psi_1^+(p)}\,, \qquad X_v(f|_{S^2}) = \frac{\partial}{\partial v}(f \circ (\psi_1^+)^{-1})|_{\psi_1^+(p)}\,.$$

Furthermore, find the integral curves of the two basis vector fields by solving the equations

$$\dot{\gamma_u}(t) = X_u(\gamma_u(t))\,, \qquad \dot{\gamma_v}(t) = X_v(\gamma_v(t))$$

for the differentiable curves

$$\gamma_u(t) = (\gamma_{u1}(t), \gamma_{u2}(t), \gamma_{u3}(t))\,, \qquad \gamma_v(t) = (\gamma_{v1}(t), \gamma_{v2}(t), \gamma_{v3}(t))\,.$$

Exercise 2. *Manifold S^2, Part 2*

Alternatively, one can define the manifold S^2 using the open sets $O_\pm = S^2 \setminus \{(0, 0, \pm 1)\}$ and stereographic projections $\phi_\pm$. The latter map a point $\vec{x} \in S^2$ to the intersection point of the x_1x_2-plane with the line through $\vec{x}$ and $(0, 0, \pm 1)$, respectively. Determine the transition function between the two charts, and show that it is smooth.

Exercise 3. *Change of Basis in Tangent and Cotangent Space*

In the chart defined by the coordinate functions x^μ, the coordinate basis for the tangent space T_p is defined by $X_\mu = \partial_\mu$, and the corresponding dual basis of the cotangent space T_p^* is given by $\mathrm{d}x^\mu$.

i) For a different chart, described by $\tilde{x}^\mu$, express the corresponding basis vectors $\widetilde{X}_\mu$ and $\mathrm{d}\tilde{x}^\mu$ in terms of X_μ and $\mathrm{d}x^\mu$, respectively. What is the transformation law of the corresponding components, i.e., writing $X = a^\mu X_\mu$ and $\omega = b_\nu \mathrm{d}x^\nu$, what is the transformation law for the coefficients a^μ and b_ν?

ii) Let $\{e_\sigma\}$ be a basis and $\{e^\sigma\}$ its dual basis. The operation of contraction C of a tensor T with respect to the ith (dual vector) and jth (vector) slots may be defined as

$$CT = T(\ldots, \underset{\substack{\uparrow \\ i}}{e^\sigma}, \ldots; \ldots, \underset{\substack{\uparrow \\ j}}{e_\sigma}, \ldots) \ , \tag{1}$$

(with summation over σ). Show that (1), and hence the operation of contraction, is independent of the choice of the basis $\{e_\sigma\}$.