Exercise 1. ***Singularities***

In the lecture the Schwarzschild solution to the vacuum Einstein equations was introduced:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega \ . \tag{1}$$

- Determine the scalar $\mathcal{R} = R_{abcd}R^{abcd}$.

  [Hint: you may use (and prove) that for a diagonal metric the non-vanishing Christoffel symbols are

  $$\Gamma^{\lambda}_{\mu\mu} = -\frac{\partial_\lambda g_{\mu\mu}}{2g_{\lambda\lambda}} \ , \qquad \Gamma^{\mu}_{\mu\lambda} = \frac{\partial_\lambda g_{\mu\mu}}{2g_{\mu\mu}} \ , \qquad \Gamma^{\mu}_{\mu\mu} = \frac{\partial_\mu g_{\mu\mu}}{2g_{\mu\mu}} \ , \tag{2}$$

  where $\mu \neq \lambda$ and there is no summation over repeated indices.]

  [Hint: The only non-vanishing components of the Riemann tensor are R_{trrt}, $R_{t\phi\phi t}$, $R_{t\theta\theta t}$, $R_{r\phi\phi r}$, $R_{r\theta\theta r}$ and $R_{\phi\theta\theta\phi}$.]

- By checking the behavior of $\mathcal{R}$ at $r = 2M$ (horizon) and $r = 0$ (origin), argue that the horizon is only a coordinate singularity while the origin is a proper singularity, *i.e.*, show that $\mathcal{R}$ diverges at $r = 0$ but not at $r = 2M$.

Exercise 2. ***Particle in Schwarzschild Background***

Consider a free particle moving radially in the Schwarzschild background with initial position r_0 ($r_0 \geq 2M$) and initial radial velocity $\dot{r}|_{r=r_0} = 0$.

- Show that the particle reaches the horizon $r = 2M$ in finite proper time.
- How long does the particle take to get to the horizon from the point of view of a static observer at infinity?

Exercise 3. ***Behind the horizon***

Show that any particle behind the horizon of the Schwarzschild background, *i.e.* with $r < 2M$, must decrease its radial coordinate at a rate given by

$$\left|\frac{\mathrm{d}r}{\mathrm{d}\tau}\right| \geq \sqrt{\frac{2M}{r} - 1} \ , \tag{3}$$

independent of whether it is in geodesic motion or not.

Hence show that the maximum lifetime of any observer behind the horizon is $\tau = \pi M$, *i.e.*, any object behind the horizon will be pulled into the singularity at $r = 0$ within this proper time. Show that this maximum time is attained by a purely radial (*i.e.* $L = 0$), free (*i.e.* geodesic) motion from $r = 2M$ with $\dot{r}|_{r=2M} = 0$.

[Hint: it is possible to avoid integration. Notice that radial fall is a movement on half of a degenerated ellipse.]