Exercise 1. ***Entanglement and teleportation***

Imagine that Alice (A) has a pure state $|\psi\rangle_S$ of a system S in her lab. She wants to send that state to Bob, who lives, of course, on the Moon, but she does not trust the postwoman Eve to carry it there personally. We have seen that if Alice and Bob share an entangled state Alice can "teleport" the state $|\psi\rangle$ to the system B that Bob controls by.

Formally, we have three systems $\mathcal{H}_S \otimes \mathcal{H}_A \otimes \mathcal{H}_B$. In this exercise we will assume all three are qubits. The initial state is

$$|\psi\rangle_S \otimes \frac{1}{\sqrt{2}} \left(|00\rangle_{AB} + |11\rangle_{AB}\right), \tag{1}$$

i.e. S is decoupled from A and B and these two are fully entangled in a Bell state. We may write $|\psi\rangle_S = \alpha|0\rangle_S + \beta|1\rangle_S$.

(a) In a first step, Alice will measure systems S and A jointly in the Bell basis,

$$\tfrac{1}{\sqrt{2}} \left(|00\rangle_{SA} + |11\rangle_{SA}\right), \tfrac{1}{\sqrt{2}} \left(|00\rangle_{SA} - |11\rangle_{SA}\right), \tfrac{1}{\sqrt{2}} \left(|01\rangle_{SA} + |10\rangle_{SA}\right), \tfrac{1}{\sqrt{2}} \left(|01\rangle_{SA} - |10\rangle_{SA}\right).$$

Then Alice communicates (classicaly) the result of her measurement to Bob. What is the reduced state of Bob's system (B) for each of the possible outcomes?

(b) Depending on the outcome of the measurement by Alice, Bob may have to perform certain unitary operations on his qubit so that he recovers $|\psi\rangle$. Which operations are these?

(c) Suppose that Alice does not manage to tell Bob the outcome of her measurement. Show that in this case he does not have any information about his reduced state and therefore does not know which operation to apply in order to obtain $|\psi\rangle$.

(d) Show that this method of quantum teleportation also works for mixed states ρ_S.

(e) There is no reason why the state ρ_S cannot be entangled with some other system that Alice and Bob do not control. Consider a purification of ρ_S on a reference system R, i.e. $|\phi\rangle_{SR}$ s.t.

$$\rho_S = \mathrm{tr}_R |\phi\rangle\langle\phi|_{SR}. \tag{2}$$

Show that if you apply the quantum teleportation protocol on $\mathcal{H}_S \otimes \mathcal{H}_A \otimes \mathcal{H}_B$, not touching the reference system, the final state on $\mathcal{H}_B \otimes \mathcal{H}_R$ is $|\phi\rangle_{BR}$.

This implies that quantum teleportation preserves correlations (including entanglement) – it simply transfers it from S and R to B and R.

Exercise 2. ***Probabilistic entanglement generation using post-selection***

Suppose we have two atoms that we would like to entangle. How can we do this? In this exercise we describe one way of doing so using post-selection. Call the identical atoms A and B. We are only interested in two energy levels of each atom, which we denote by $|g\rangle$ (ground state) and $|e\rangle$ (excited state). From now on we treat them as qubits.

(a) Assume that both atoms are initially in the excited state, $|\Psi_{\text{in}}\rangle_{AB} = |e\rangle_A |e\rangle_B$. Due to spontaneous emission the atoms may decay to the ground state. Taking the electromagnetic field (emf) into account, what is the state of the total system ($A \otimes B \otimes \text{emf}$) after some time?

Hint: Assume that the probability for each atom to decay during this time is ε and that this happens independently for both atoms. Also, split the emf into the relevant modes.

(b) If you can make arbitrary measurements on the emf, i.e. on the emitted photons if there are any, find a measurement that, if the right outcome is obtained, leaves the atoms in a maximally entangled state. What is the probability that this outcome occurs?

Remark: This is called post-selection. Depending on the outcome of the measurement we know that the atoms are entangled (and we can do interesting things with them) or not. In the latter case the protocol would be aborted.

(c) Can you come up with an experimental setup to perform the proposed measurement?

Exercise 3. ***Gibbs paradox***

(a) Consider a container consisting of two compartments separated by a removable wall (left figure below). The two compartments are filled with an ideal gas at the same pressure and temperature. Assume first that you are not aware of any process that distinguishes the gas in the left compartment from that in the right one. By how much does the entropy of the container change when you remove the wall?

(b) Assume now that you discover two materials, A and B, and that you find that the gas in the left compartment can pass through A but not through B, and that the gas in the right compartment can pass through B but not through A. The existence of two materials with those characteristics shows, in particular, that the two compartments are actually filled with two different types of gas (right figure below). By how much does the entropy of the container now change when removing the wall? Carry out this calculation both in the framework of Phenomenological Thermodynamics and in Statistical Mechanics.

Hint: In order to calculate the entropy change in the framework of Phenomenological Thermodynamics you should try to find a reversible process resulting in the same state change and use $dS = \delta Q_{\text{rev}}/T$, where Q_{rev} is the reversibly exchanged heat and T is the temperature of the heat bath.

(c) Explain the different outcomes of the entropy changes in the above scenarios.