Fourier transforms

Fourier transforms on a lattice of size L with lattice spacing a and $N = L/a$ sites. When x and k are discrete they are given by $x = ma$ and $k = 2\pi n/L$. For the finite discrete system n and m are restricted to the range $[-N/2, N/2[$ while otherwise they span all integers. (because in those cases $N \to \infty$)

$0 < a < L < \infty$

$$\widetilde{f}(k_n) = \mathcal{F}[f(x_m)] = \sum_{m=-N/2}^{N/2-1} f(x_m) e^{-ik_n x_m} \qquad f(x_l) = \mathcal{F}^{-1}[\widetilde{f}(k_n)] = \frac{1}{N} \sum_{n=-N/2}^{N/2-1} \widetilde{f}(k_n) e^{ik_n x_l}$$

$$X_m = \frac{1}{N} \sum_{n=-N/2}^{N/2-1} e^{ik_n x_m} = \sum_l \delta_{m,lN} = \delta_{m,0}$$

$$K_{k_n} = \frac{1}{N} \sum_{m=-N/2}^{N/2-1} e^{-ik_n x_m} = \sum_l \delta_{k,\frac{2\pi l}{L}} = \delta_{k,0}$$

$0 \leftarrow a < L < \infty$

$$\widetilde{f}(k_n) = \mathcal{F}[f(x)] = \frac{1}{L} \int_{-L/2}^{L/2} dx f(x) e^{-ik_n x} \qquad f(x) = \mathcal{F}^{-1}[\widetilde{f}(k_n)] = \sum_{n=-\infty}^{\infty} \widetilde{f}(k_n) e^{ik_n x}$$

$$X(x) = \frac{1}{L} \sum_{n=-\infty}^{\infty} e^{ik_n x} = \sum_l \delta(x - lL) = \delta(x)$$

$$K_{k_n} = \frac{1}{L} \int_{-L/2}^{L/2} dx e^{-ik_n x} = \delta_{k,0}$$

$0 < a < L \to \infty$

$$\widetilde{f}(k) = \mathcal{F}[f(x_m)] = \sum_{m=-\infty}^{\infty} f(x_m) e^{-ikx_m} \qquad f(x_l) = \mathcal{F}^{-1}[\widetilde{f}(k)] = \frac{a}{2\pi} \int_{-\pi/a}^{\pi/a} dk \widetilde{f}(k) e^{ikx_l}$$

$$X_m = \frac{a}{2\pi} \int_{-\pi/a}^{\pi/a} dk e^{ikx_m} = \delta_{m,0}$$

$$K(k) = \frac{a}{2\pi} \sum_{m=-\infty}^{\infty} e^{-ikx_m} = \sum_l \delta\left(k - \frac{2\pi l}{L}\right) = \delta(k)$$

$0 \leftarrow a < L \to \infty$

$$\widetilde{f}(k) = \mathcal{F}[f(x)] = \int_{-\infty}^{\infty} dx f(x) e^{-ikx} \qquad f(x) = \mathcal{F}^{-1}[\widetilde{f}(k)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \widetilde{f}(k) e^{ikx}$$

$$X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx e^{ikx} = \delta(x)$$

$$K(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{-ikx} = \delta(k)$$