Exercise 1. Dirac δ function reminder

Lernziel: The Dirac delta function appears frequently when performing Fourier transforms. In quantum mechanics Fourier transforms allow us to switch basis from real space to momentum space and vice versa. As we will often need to do this we will encounter delta functions on a regular basis and should familiarise ourselves with them.

The Dirac delta function may be defined by:

$$\delta(x) = 0 \text{ if } x \neq 0 \qquad \text{and} \qquad \int_{-\infty}^{\infty} \delta(x) dx = 1$$

1.1. *Representations*

Possibly the simplest way to picture this is the top hat function in the limit $a \to 0$:

$$f(x) = \begin{cases} 0 & \text{if } |x| > a \\ \frac{1}{2a} & \text{if } |x| \leq a \end{cases}$$

However there are many other representations. For each of the following functions:

(a) $f(x) = \alpha e^{-x^2/\sigma^2}$ (b) $g(x) = \frac{\alpha\sigma}{x^2+\sigma^2}$ (c) $h(x) = \alpha\sigma \left(\frac{\sin(x/\sigma)}{x}\right)^2$

show that in the limit $\sigma \to 0$ they become delta functions and find the corresponding α to normalise them properly.

1.2. *Properties*

The key property of the Dirac delta function is the following:

$$\int_{-\infty}^{\infty} \delta(x - x_0) f(x) dx = f(x_0)$$

With this starting point in mind evaluate:

$$\int_{-\infty}^{\infty} \delta(f(x)) dx$$

Find explicit expressions for the following two cases:

(a) $f(x) = ax$ (b) $f(x) = x^2 - x_0^2$

Exercise 2. Fourier transforms reminder

> *Lernziel: Fourier transforms play a central role in quantum mechanics, allowing us to switch between different basis. Depending on the size of the system being considered and whether it is continuous or discrete in space we have to use different transforms. Here we familiarise ourselves with this concept.*

Consider a 1D system of length L with sites spaced by a, as in figure 1.

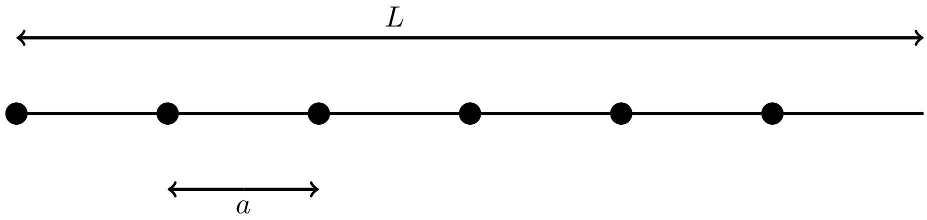

Figure 1: A lattice with lattice constant a involving $N = L/a$ sites at locations $x_m = am$.

Give an example of a physical system which may be well described by the four following limits (ignore the fact we are in 1D):

(a) $0 < a < L < \infty$

(b) $0 \leftarrow a < L < \infty$

(c) $0 < a < L \to \infty$

(d) $0 \leftarrow a < L \to \infty$

2.1. *More representations of δ functions and Kroenicker δs* For case (a), evaluate the sums

$$X_m = \frac{1}{N} \sum_{n=-N/2}^{N/2-1} e^{ik_n x_m} \qquad \text{and} \qquad K_n = \frac{1}{N} \sum_{m=-N/2}^{N/2-1} e^{-ik_n x_m}$$

with $k_n = 2\pi n/L$. Extend the expressions X_m and K_n to the cases (b), (c) and (d) and evaluate them. What prefactors must you use to have dimensionless and normalised results at the end? Note the relation to Poisson's formula.

Hint: Recall the formula for a geometric series and numbers on the unit circle in the complex plane:

$$\sum_{n=0}^{N-1} z^n = \frac{1 - z^N}{1 - z} \qquad \textit{and} \qquad \sum_{n=0}^{N-1} e^{2\pi i n/N} = 0$$

You may use the following representation of a δ function:

$$\lim_{k \to \infty} \frac{sin(kx)}{x} = \pi\delta(x)$$

2.2. ***Fourier transforms and their inverses***

We now define the Fourier transforms and their inverses as follows for the four situations:

(a)

$$\widetilde{f}(k_n) = \mathcal{F}[f(x_m)] = \sum_{m=-N/2}^{N/2-1} f(x_m)e^{-ik_n x_m}$$

$$\widehat{f}(x_l) = \mathcal{F}^{-1}[\widetilde{f}(k_n)] = \frac{1}{N}\sum_{n=-N/2}^{N/2-1} \widetilde{f}(k_n)e^{ik_n x_l}$$

(b)

$$\widetilde{f}(k_n) = \mathcal{F}[f(x)] = \frac{1}{L}\int_{-L/2}^{L/2} dx f(x)e^{-ik_n x}$$

$$\widehat{f}(x) = \mathcal{F}^{-1}[\widetilde{f}(k_n)] = \sum_{n=-\infty}^{\infty} \widetilde{f}(k_n)e^{ik_n x}$$

(c)

$$\widetilde{f}(k) = \mathcal{F}[f(x_m)] = \sum_{m=-\infty}^{\infty} f(x_m)e^{-ikx_m}$$

$$\widehat{f}(x_l) = \mathcal{F}^{-1}[\widetilde{f}(k)] = \frac{a}{2\pi}\int_{-\pi/a}^{\pi/a} dk \widetilde{f}(k)e^{ikx_l}$$

(d)

$$\widetilde{f}(k) = \mathcal{F}[f(x)] = \int_{-\infty}^{\infty} dx f(x)e^{-ikx}$$

$$\widehat{f}(x) = \mathcal{F}^{-1}[\widetilde{f}(k)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \widetilde{f}(k)e^{ikx}$$

For each of these explicitly show $\mathcal{F}^{-1}[\mathcal{F}[f]] = f$. Use your results from exercise 2.1.

Exercise 3. Green's function reminder

Lernziel: Green's functions allow us to solve non-homogeneous boundary value problems, which are frequent in physics. Here we evaluate the Green's function for a simple linear operator to refresh our memories.

3.1. Evaluate the Green's function for the linear operator $L_x = \partial_x^2 - k_0^2$.

3.2. Find the solution of the driven equation $L_x f(x) = S(x)$ for a given source $S(x)$.

3.3. * Analytically continue your result in 3.1 to the operator $L_x = \partial_x^2 + k_0^2$. What physics does this represent?

Exercise 4. Physics III reminder

Lernziel: Matter may exhibit wavelike behaviour, for example an electron beam may be diffracted, just like a beam of light. If the wavelength is too small then the dynamics will be well described by classical mechanics. Here we will calculate the wavelength of various objects and consider their implications.

4.1. Starting from the de Broglie hypothesis $p = h/\lambda$ calculate the wavelength of massless and massive particles with kinetic energy E_{kin}.

4.2. Evaluate the wavelengths of the following objects with $E_{\text{kin}} = 1\text{eV}$:

(a) a photon

(b) an electron

(c) a water molecule

(d) a 1 kg football

Comment on the results.

Exercise 5. Physics III reminder

Lernziel: It is often possible to obtain the correct physical behaviour without performing any detailed calculations. Here we will find a rough approximation for the ionisation temperature of hydrogen without solving its Hamiltonian.

5.1. Write down the x, p uncertainty principle and the Hamiltonian for a hydrogen atom. Using all the available parameters estimate the binding energy of hydrogen.

5.2. Use your result to estimate the ionising temperature of hydrogen. What is the state of hydrogen in the sun?