Exercise 1. Dirac δ function reminder

1.1. *Representations*

We find α as a function of σ in order for the three functions to be normalised and then take the limit as $\sigma \to 0$ to check the values of the function for $x \neq 0$:

(a) The Gaussian integral is a standard integral, giving us:

$$\int_{-\infty}^{\infty} \alpha e^{-(x/\sigma)^2} dx = \alpha\sigma\sqrt{\pi} = 1 \quad \Rightarrow \quad \lim_{\sigma\to 0} \frac{e^{-(x/\sigma)^2}}{\sigma\sqrt{\pi}} = \delta(x)$$

(b) The Lorentzian is also a standard integral, giving us:

$$\int_{-\infty}^{\infty} \frac{\alpha\sigma}{x^2+\sigma^2} dx = \alpha\pi = 1 \quad \Rightarrow \quad \lim_{\sigma\to 0} \frac{\sigma}{\pi}\frac{1}{x^2+\sigma^2} = \delta(x)$$

(c) The sinusoidal integral is more tricky. First we use a trigonometric identity and write it as the real part of a complex valued function:

$$\int_{-\infty}^{\infty} \alpha\sigma\frac{\sin^2(x/\sigma)}{x^2} dx = \int_{-\infty}^{\infty} \alpha\sigma\frac{1-\cos(2x/\sigma)}{2x^2} dx = \mathrm{Re}\int_{-\infty}^{\infty} \alpha\sigma\frac{1-e^{2ix/\sigma}}{2x^2} dx$$

Now we may close the contour in the upper half of the complex plane and take half the residue at $x = 0$:

$$\mathrm{Re}\int_{-\infty}^{\infty} \alpha\sigma\frac{1-e^{2ix/\sigma}}{2x^2} dx = \mathrm{Re}\ 2\pi i\alpha\frac{-2i}{4} = \alpha\pi = 1 \quad \Rightarrow \quad \lim_{\sigma\to 0}\frac{\sigma}{\pi}\frac{\sin^2(x/\sigma)}{x^2} = \delta(x)$$

All three of these functions are 0 for $x \neq 0$ and are normalised to 1, therefore they are all representations of the δ function.

1.2. *Properties*

As the δ function is 0 everywhere except when $f = 0$ we may concentrate on the region around the roots x_i of $f(x)$ and expand $f(x) = (x - x_i)f'(x_i)$. Note that higher order roots are not well defined.

$$\int_{-\infty}^{\infty} \delta(f(x))dx = \sum_{x_i}\int_{x_i-\epsilon}^{x_i+\epsilon} \delta(f(x))dx = \sum_{x_i}\int_{x_i-\epsilon}^{x_i+\epsilon} \delta((x-x_i)f'(x_i))dx$$

Next we use a substitution $y = f'(x_i)(x - x_i)$ to get:

$$\sum_{x_i} \int_{x_i-\epsilon}^{x_i+\epsilon} \delta((x-x_i)f'(x_i))dx = \sum_{x_i} \frac{1}{f'(x_i)} \int_{-f'(x_i)\epsilon}^{f'(x_i)\epsilon} \delta(y)dy = \sum_{x_i} \frac{\text{sgn}(f'(x_i))}{f'(x_i)} = \sum_{x_i} \frac{1}{|f'(x_i)|}$$

Evaluating it explicitly for the two cases we obtain:

(a) $\frac{1}{|a|}$

(b) $\frac{1}{|x_0|}$

Exercise 2. Fourier transforms reminder

Physical examples may be:

(a) A small crystal

(b) A string of length L

(c) The bulk of a very large crystal

(d) Vacuum

2.1. *More representations of* δ

We concentrate on X_m we have two cases, first $m = lN$ with l some integer:

$$X_{lN} = \frac{1}{N} \sum_{n=-N/2}^{N/2-1} e^{i2\pi nlL/L} = \frac{1}{N} \sum_{n=-N/2}^{N/2-1} 1 = \frac{N}{N} = 1$$

We then have the second case with all other m. Here we use the fact that the sum of evenly spaced numbers on the unit circle in the complex plane is 0:

$$X_{m\neq lN} = \frac{1}{N} \sum_{n=-N/2}^{N/2-1} e^{i2\pi nma/L} = \frac{1}{N} \sum_{n=-N/2}^{N/2-1} e^{i2\pi nm/N} = 0$$

Thus we know both X_m and K_n:

$$X_m = \sum_l \delta_{m,lN} = \delta_{m,0} \qquad \text{and} \qquad K_n = \sum_l \delta_{n,lN} = \delta_{n,0}$$

As both n and m have a restricted range the terms with $|l| > 0$ are truncated.

We can now extend this to the other cases. For (b) we have:

$$X_m = \frac{1}{N} \sum_{n=-N/2}^{N/2-1} e^{ik_n x_m} \rightarrow X(x) = \alpha \sum_{n=-\infty}^{\infty} e^{ik_n x}$$

Where α is some normalisation we will determine. For $x = lL$ this sum is clearly infinite while for any other x it will be 0. This reminds us of a sum of δ functions. We will now check the normalisation:

$$\int_{-L/2}^{L/2} dx \sum_{n=-\infty}^{\infty} e^{ik_n x} = L + \sum_{n=-\infty}^{-1} \int_{-L/2}^{L/2} dx e^{ik_n x} + \sum_{n=1}^{\infty} \int_{-L/2}^{L/2} dx e^{ik_n x}$$

$$= L + \sum_{n=-\infty}^{-1} \frac{e^{i\pi n} - e^{-i\pi n}}{ik_n} + \sum_{n=1}^{\infty} \frac{e^{i\pi n} - e^{-i\pi n}}{ik_n} = L$$

We now know what the normalisation should be and can write down our final solution:

$$X(x) = \frac{1}{L} \sum_{n=-\infty}^{\infty} e^{ik_n x} = \sum_l \delta(x - lL) = \delta(x)$$

The other sum becomes an integral with $dx = a$ and remains properly normalised:

$$K_n = \frac{1}{N} \sum_{m=-N/2}^{N/2-1} e^{-ik_n x_m} \to K_n = \frac{1}{N}\frac{N}{L} \int_{-L/2}^{L/2} dx e^{-ik_n x} = \frac{1}{L} \int_{-L/2}^{L/2} dx e^{-ik_n x}$$

For $k_n = 0$ this is integral is simply unity while in all other cases it is 0:

$$\frac{1}{L} \int_{-L/2}^{L/2} dx e^{-i2\pi nx/L} = \frac{e^{-i\pi n} - e^{i\pi n}}{ik_n} = 0$$

Giving us the final result:

$$K_n = \frac{1}{L} \int_{-L/2}^{L/2} dx e^{-ik_n x} = \delta_{n,0}$$

For (c) we go through the same steps to obtain:

$$X_m = \frac{a}{2\pi} \int_{-\pi/a}^{\pi/a} dk e^{ikx_m} = \delta_{m,0}$$

$$K(k) = \frac{a}{2\pi} \sum_{m=-\infty}^{\infty} e^{-ikx_m} = \sum_l \delta(k - 2\pi l/L) = \delta(k)$$

For (d)we obtain two integrals:

$$X(x) = \alpha \int_{-\infty}^{\infty} dk e^{ikx} \qquad \text{and} \qquad K(k) = \alpha \int_{-\infty}^{\infty} dx e^{-ikx}$$

Both of them will obviously give the same result so we will concentrate on $X(x)$, we see that for $x = 0$ it is infinite while everywhere else it is 0 again reminding us of a delta function. To check we perform the integral from $-R$ to R and take the limit as R goes to infinity.

$$X(x) = \lim_{R\to\infty} \alpha \int_{-R}^{R} dk e^{ikx} = \alpha \lim_{R\to\infty} \frac{2sin(kR)}{k} = 2\pi\alpha\delta(x)$$

We are now in a position to write down the final two terms:

$$X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ikx} = \delta(x) \qquad \text{and} \qquad K(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx e^{-ikx} = \delta(k)$$

Each time we had an integral we obtained a single δ, when we had a sum we obtained a sum of δs which was truncated by the finite size or the discrete nature of the system. These truncated terms are similar to the ones used in proving the Poisson summation formula. In k space the range outside of $[-\pi/a, \pi/a]$ where these extra terms appear is useful when dealing with properties of solids, and will be called Brillouin zones.

2.2. *Fourier transforms and their inverse*

We simply write $\widehat{f} = \mathcal{F}^{-1}[\mathcal{F}[f]]$ in each case and check that $\widehat{f} = f$:

(a)

$$\widehat{f}(x_l) = \sum_{m=-N/2}^{N/2-1} f(x_m) \frac{1}{N} \sum_{n=-N/2}^{N/2-1} e^{ik_n(x_l - x_m)} = \sum_{m=-N/2}^{N/2-1} f(x_m)\delta_{m,l} = f(x_l)$$

(b)

$$\widehat{f}(x) = \int_{-L/2}^{L/2} dx' f(x') \frac{1}{L} \sum_{n=-\infty}^{\infty} e^{ik_n(x-x')} = \int_{-L/2}^{L/2} dx' f(x')\delta(x'-x) = f(x)$$

(c)

$$\widehat{f}(x_l) = \sum_{m=-\infty}^{\infty} f(x_m) \frac{a}{2\pi} \int_{-\pi/a}^{\pi/a} dk e^{ik(x_l - x_m)} = \sum_{m=-\infty}^{\infty} f(x_m)\delta_{m,l} = f(x_l)$$

(d)

$$\widehat{f}(x) = \int_{-\infty}^{\infty} dx' f(x') \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ik(x-x')} = \int_{-\infty}^{\infty} dx' f(x')\delta(x'-x) = f(x)$$

Exercise 3. Green's function reminder

3.1. ***1*** The Green's function is given by $(\partial_x^2 - k_0^2)G(x) = \delta(x)$. We can Fourier transform this to obtain:

$$-(k^2 + k_0^2)\widetilde{G}(k) = 1$$

And therefore

$$G(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ikx}}{-(k^2 + k_0^2)}$$

Which when we evaluate it in turn gives

$$G(x) = \frac{e^{-k_0|x|}}{-2k_0}$$

3.2. ***1*** We now have the differential equation $(\partial_x^2 - k_0^2)f(x) = S(x)$ for some source $S(x)$. We proceed as in the previous section:

$$-(k^2 + k_0^2)\widetilde{f}(k) = \widetilde{S}(k)$$

$$\widetilde{f}(k) = -\frac{\widetilde{S}(k)}{k^2 + k_0^2}$$

$$f(x) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\widetilde{S}(k)e^{ikx}}{k^2 + k_0^2} dk = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{S(x')e^{ik(x-x')}}{k^2 + k_0^2} dk dx'$$

3.3. ***1***

By simply using the substitution $k_0 \to ik_0$ we obtain:

$$G(x) = i\frac{e^{-ik_0|x|}}{2k_0}$$

Which is waves being excited at $x = 0$ and propagating outwards.

Exercise 4. Physics III reminder

We start with the de Broglie hypothesis $p = h/\lambda$ and first use the dispersion relation of a massless particle:

$$E_{kin} = pc \qquad \Rightarrow \qquad \lambda = \frac{hc}{E}$$

We now repeat the same process for a massive particle:

$$E_{kin} = \frac{p^2}{2m} \qquad \Rightarrow \qquad \lambda = \frac{h}{\sqrt{2mE}}$$

With these two expressions we now evaluate explicitly for various objects with $E_{kin} = 1eV$

(a) $\lambda_\gamma = 1.2 \cdot 10^{-6}m$

(b) $\lambda_e = 1.2 \cdot 10^{-9}m$

(c) $\lambda_{H_2O} = 6.7 \cdot 10^{-12}m$

(d) $\lambda_{football} = 1.2 \cdot 10^{-24}m$

We see that larger masses lead to smaller wavelengths a football is clearly not to be considered as behaving quantum mechanically as we will never be able to resolve it on such a small scale.

Exercise 5. Physics III reminder

The Hamiltonian for a hydrogen atom and the uncertainty principle are given by:

$$H = \frac{p^2}{2m} - \frac{e^2}{4\pi\epsilon_0 r} \qquad \text{and} \qquad \Delta x \Delta p \geq \frac{\hbar}{2}$$

From the uncertainty principle we get $\Delta x \sim r$, $\Delta p \sim p$ and $pr \sim \hbar/2$. Using the Virial theorem we can now estimate the binding energy of Hydrogen:

$$\frac{p^2}{m} \sim \frac{2pe^2}{4\pi\epsilon_0\hbar} \qquad \Rightarrow \qquad p \sim \frac{e^2 m}{\epsilon_0 h} \qquad \Rightarrow \qquad E \sim \frac{e^4 m}{2\epsilon_0^2 h^2} \approx 54eV$$

This is too large but means we will not be astonished when we find it is actually $\sim 13.6eV$. By setting $E = k_B T$ it also gives a temperature $T \sim 6 \cdot 10^5 K$, the real result is about $T \sim 10^4 K$. If we compare this to the temperatures in the centre of the sun ($T \sim 10^7 K$) we see that Hydrogen in the sun is a plasma.