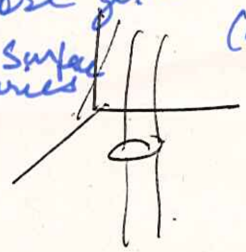
Flux line

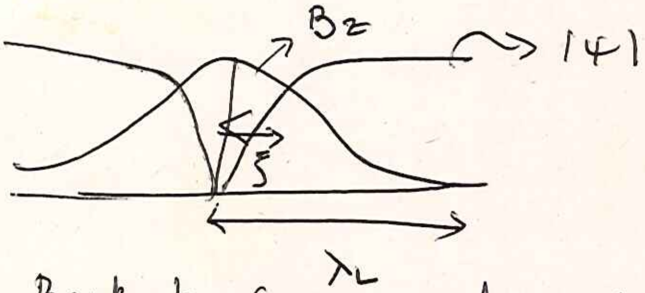
$$\Phi_0 = \frac{h}{2e} = 2\pi\sqrt{2}\, B_c \frac{\lambda_L^2}{k} \equiv 2\pi\sqrt{2}\, B_c \lambda_L \xi .$$

On the homogeneous state with $\varphi = 1$, turn up mag field till first normal state island appears → suppose germ is along z. (along field lines)
Since surface energy is <0 tendency to form many surface boundaries
But flux connected with normal states in SC are quantized! $\Phi \geq n\Phi_0$.



Consider a flux line along $\hat{z}$ with flux Φ_0.

$$\Psi = |\psi| e^{i\theta} \qquad \vec{B} = \hat{z}\, B(\rho) \qquad \rho^2 = x^2 + y^2$$



Back to GLE. Assume $\lambda_L \gg \xi$ $\quad k \gg 1$ &

consider the regions $\rho \gg \xi$ where $|\psi|$ is constant.

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j}_s \quad [\text{Ampère's law}]$$

$$= \frac{1}{\lambda_L^2}\left[\frac{\Phi_0}{2\pi}\vec{\nabla}\theta - \vec{A}\right]$$

$$\Rightarrow \quad \vec{A} + \lambda_L^2\, \vec{\nabla} \times \vec{B} = \frac{\Phi_0}{2\pi}\vec{\nabla}\theta . \qquad \text{—(1)}$$

Integrate (1) along circle around flux line with $r \gg \xi$.
& using Stokes $\oint d\vec{s} \cdot \vec{A} = \int d^2 r \cdot (\vec{\nabla} \times \vec{A})$

$$\Rightarrow \int d^2 r \cdot \vec{B} + \lambda \oint d\vec{s} \cdot (\vec{\nabla} \times \vec{B}) = \Phi_0 .$$ because θ must increase by 2π around a fluxoid.

~~Now, consider~~ $\cancel{\xi << \rho << \lambda_L}$.

Reapplying Stokes theorem, for all $r >> \xi$ we have

$$\int d^2\vec{n}\cdot\left(\vec{B} + \lambda_L^2\, \vec{\nabla}\times\vec{\nabla}\times\vec{B}\right) = \int d^2\vec{n}\cdot\left(\vec{B} - \lambda_L^2\, \nabla^2\vec{B}\right)$$

$$= \Phi_0.$$

Since the ~~flux~~ does not change if we vary the area of integration, then

$$\vec{B} - \lambda_L^2\, \nabla^2\vec{B} = \vec{\Phi}_0\, \hat{z}\ \ \delta^2(r) \qquad (2)$$

$$\vec{\Phi}_0 = \Phi_0\, \hat{z}$$

In cylindrical coords

$$\cancel{\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}}$$

$$\Rightarrow \cancel{B'' + \frac{1}{r}B' - \frac{1}{\lambda_L^2}B} = \Phi_0\, \delta^2(r)$$

Bessel eqn:

$$B(r) = \frac{\Phi_0}{2\pi\lambda_L^2} K_0\left[\frac{r}{\lambda_L}\right]$$

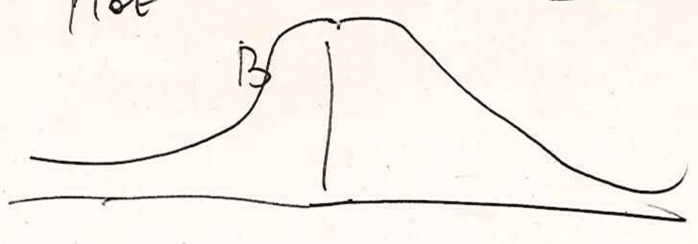
$$\cancel{B(r) = \frac{\Phi_0}{2\pi}}$$

$$\propto \ln\frac{\lambda_L}{r} \qquad r << \lambda_L$$

$$\cancel{B_k = \frac{\Phi_0}{1+\lambda^2 \vec{k}^2}}$$

$$\propto \left(\frac{\lambda_L}{r}\right)^{1/2} e^{-\frac{r}{\lambda_L}} \qquad r >> \lambda$$

$\cancel{B(r) =}$ Plot



current around the vortex.

$$\vec{j}_s \propto \vec{\nabla}\times\vec{B} \propto \frac{dB(r)}{dr}$$

$$\approx \begin{cases} \frac{\Phi_0}{\lambda_L^2 r} & r \ll \lambda_L \\ \frac{\Phi_0}{\lambda_L^3}\sqrt{\frac{\lambda_L}{r}}\, e^{-r/\lambda_L} & r \gg \lambda_L \end{cases}$$

For $r \to 0$ all divergences are cut off becase

the δ fn. width is $O(\xi)$.

Maximal current density in vortex core j_c !

Energy of vortex line [field energy + KE of supercurrent]

$$\mathcal{E} = \int d^3V \left[\frac{\vec{B}^2}{2\mu_0} + \frac{m}{2}|\psi|^2 v^2 \right] \qquad j_s = -q|\psi|^2 v$$

$\int d^2r$

$$\frac{m|\psi|^2}{2}\frac{j_s^2}{q^2|\psi|^4}$$

$$\frac{m\, j_s^2}{2q^2|\psi|^2} \Big]$$

$$\frac{m}{2q^2|\psi|^2}\frac{1}{\mu_0^2}(\vec{\nabla}\times\vec{B})^2 \Big]$$

$$= \frac{1}{2\mu_0}\int dV \left[\frac{\vec{B}^2}{\mu_0} + \lambda_L^2 (\vec{\nabla}\times\vec{B})^2 \right] \qquad \lambda_L^2 = \frac{m}{q^2|\psi|^2\mu_0}$$

$$= \cancel{\frac{1}{2\mu_0}\int}$$

$$(\vec{\nabla}\times\vec{B})^2 = \vec{B}\cdot(\vec{\nabla}\times\vec{\nabla}\times\vec{B}) + \vec{\nabla}\cdot(\vec{B}\times\vec{\nabla}\times\vec{B}).$$

$$\mathcal{E} \propto \int \vec{B}\cdot[\vec{B} + \lambda_L^2 \vec{\nabla}\times\vec{\nabla}\times\vec{B}] + \lambda_L^2 \int \vec{\nabla}\cdot(\vec{B}\times\vec{\nabla}\times\vec{B})$$

$$= \frac{1}{2\mu_0}\left[\int \cdots \quad + \lambda_L^2 \oint (\vec{B}\times\vec{\nabla}\times\vec{B})\cdot d\vec{S}\right]$$

$\rightarrow 0$ because farther away from core, $\vec{B} + \vec{\nabla}\times\vec{B} \rightarrow 0$.

Using (21),

$$\mathcal{E} \propto \int \underline{\vec{B}\cdot\vec{\Phi}_0}\, \delta^2(r)$$

Energy of vortex of length L is

$$\mathcal{E} = \bar{\mathcal{E}} L = \frac{\Phi_0 B(0) L}{2\mu_0} \simeq \frac{\Phi_0 B(\xi) L}{2\mu_0}.$$

$$\bar{\mathcal{E}} \approx \frac{\Phi_0^2}{4\pi\mu_0\lambda_L^2} \ln\frac{\lambda_L}{\xi} = \frac{\Phi_0^2}{4\pi\mu_0\lambda^2} \ln\kappa.$$

for n quanta vortex $\bar{\mathcal{E}} \propto n^2$ because $\Phi_0 \rightarrow n\Phi_0$

Total energy is minimum for flux lines containing one fluxoid

while for n flux lines $\bar{\mathcal{E}} \propto n$.

Interaction between vortex / flux lines

Consider 2

[So why not create many vortices inst all the time ?].

Mag field satisfies

$$\vec{B} + \lambda_L^2 \, \vec{\nabla} \times \vec{\nabla} \times \vec{B} = \Phi_0 \left[\delta(r - r_1) + \delta(r - r_2) \right]$$

$\Rightarrow$ linear superpos of fields [Green's fn]

$$\vec{B} = \vec{B}_V(\bar{r} - \bar{r}_1) + \vec{B}_V(\bar{r} - \bar{r}_2)$$

Again sub in GLf

$$\bar{e} = \frac{\Phi_0}{2\mu_0} \left[B(\bar{r}_1) + B(\bar{r}_2) \right] = 2\varepsilon + \frac{\Phi_0}{2\mu_0} B_{12}$$

$$= \frac{\Phi_0}{2\mu_0} \left[\underbrace{2B_1(r_1)}_{\text{sum of 2 ind. vortex energies}} + \underbrace{B_1(r_2)}_{\text{interaction}} \right]$$

$$\bar{B}(\bar{r}_1) = B(0) + B_{12}(r_1 - r_2)$$

$$B(r_2) = B(0) +$$

$\Rightarrow$ interaction energy

$$U(\underbrace{r_1 - r_2}_{x}) = \frac{\Phi_0}{\mu_0} B_{12}(x) \Rightarrow \frac{\Phi_0^2}{2\pi \lambda_L^2} K_0\left(\frac{|x|}{\lambda_L}\right)$$

repulsive for vortices of same sign

force $f = -\frac{dU}{dx} = -\frac{\Phi_0}{\mu_0} \frac{dB_{12}}{dx} \rightarrow$ repulsive force or not

But Maxwell eqns

$$\Rightarrow \frac{dB_{12}}{dx} = \mu_0 J_{12}(x)$$

Magnus force

$$\Rightarrow f = \Phi_0 (J \times \hat{z})$$

Vortex of same sign of circulation repel each other.
Lorentz force acts on vortices under current.

→ this is hydrodynamic force in nature.

Using $J = n_s e v_s$

$$f \propto \vec{v}_s \times \vec{ev}$$

What field makes it favorable for the creation of the first vortex?

Back to Gibbs.

$$G = F - \frac{B\, B_{ext}}{\mu_0 \cdot \cancel{(22)}}$$

$F = \bar{\varepsilon}\, \frac{B}{\Phi_0}$ [no of vortices or vortex density]

$$G = B\left[\frac{\bar{\varepsilon}}{\Phi_0} - \frac{B_{ext}}{\mu_0}\right]$$

for $\frac{B_{ext}}{\mu_0} > \frac{\bar{\varepsilon}}{\Phi_0} \Rightarrow B_{c1} = \ldots$

$$\frac{\bar{\varepsilon}}{\Phi_0} = \frac{\Phi_0 \ln \kappa}{4\pi \mu_0 \lambda_L^2} = \frac{B_{c1}}{\mu_0}.$$

$$\boxed{B_{c1} = \frac{\Phi_0 \ln \kappa}{4\pi \lambda_L^2}}$$

Vortex density

$\bar{\mathcal{E}} = \frac{\Phi_0^2 \ln\kappa}{\mu_0 4\pi \lambda_L^2}$ $\quad B_{c1} = \frac{\bar{\mathcal{E}}\mu_0}{\Phi_0}$.

Minimize Gibbs

$$\frac{\partial G}{\partial B} = \frac{\partial}{\partial B}\left[\frac{B}{\Phi_0}\left[\mathcal{E}_l + \frac{1}{2}\sum_{j\neq 0} U_{0j}\right] - \frac{B H}{\mu_0}\right] = 0$$

$H \to$ ext field.

$$\frac{\mathcal{E}_L}{\Phi_0} - \frac{2B}{\mu_0} - \frac{H}{\mu_0} \cdots$$

$\frac{\mathcal{E}_L}{\Phi_0} = \frac{B_{c1}}{\mu_0} =$

$$\boxed{\frac{H - H_{c1}}{\mu_0} = \frac{\partial}{\partial B}\frac{1}{2}\frac{B}{\Phi_0}\sum_{j\neq 0} U_{0j}}$$

[Get nice picture]

B_{c1}

$= \cdots$

$\frac{B_{c1}}{\mu_0} - H$

Low fields. $B \ll H_{c2}$:

Vortex x^{ns} are exponentially weak $\to$ just consider nearest neighbor vortices.

Then maximize the intervortex distance.

$\Rightarrow$ Triangular lattice (hexagonal?)

$$U_{ij} \propto e^{-\frac{r_{ij}}{\lambda_L}} = e^{-\frac{a_0}{\lambda_L}}$$

at $B = B_{c1}$

$$\frac{B_{c1}}{\Phi_0} = \frac{\ln\kappa}{4\pi\lambda_L^2}$$

$$a_0 = \left(\frac{2}{\sqrt{3}}\right)^{1/2}\sqrt{\frac{\Phi_0}{B}}$$

$$\boxed{a_1^2\frac{\sqrt{3}}{2} = \frac{4\pi\lambda^2}{\ln\kappa}}.$$

$\downarrow$ $a = \sqrt{\frac{8\pi}{\sqrt{3}\ln\kappa}}\,\lambda$

$\gtrsim \lambda$

Intermediate fields $\quad \frac{\Phi_0}{\lambda_L^2} \ll B \ll \frac{\Phi_0}{\xi^2}$.

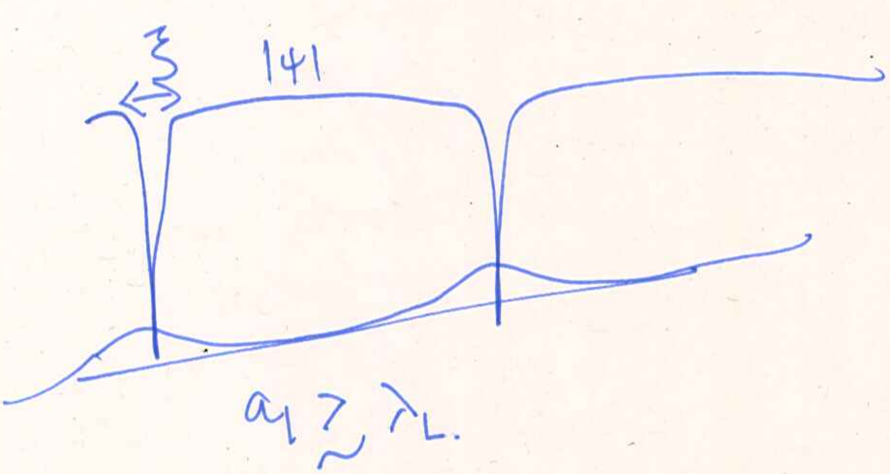
vortex cores do not overlap. but x^n long ranged!

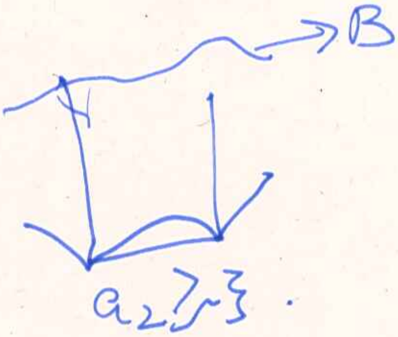
High fields $\to$ vortex cores overlap & order parameter is reduced!

B_{c_2} : $$B_{c_2} = \frac{B_{c_1} 2\kappa^2}{\ln \kappa}.$$

$$a_2 = a\sqrt{\frac{4\pi}{\sqrt{3}}\frac{\lambda_L}{\kappa}} = \sqrt{\frac{4\pi}{\sqrt{3}}}\,\xi \gtrsim \xi .$$

vortices touch & overlap!





Mag. $$B_0 = B_{ext} - \mu_0 M .$$

$$dF = M\,dB_{ext}$$

check this expression from older notes