Introduction to Superconductivity
Exercise Sheet 9

Discussion date: 25 November 2015

Exercise 1: Dilute Bose gas

This exercise serves the purpose of getting acquainted again with second quantization and the Bogoliubov transformation. It also serves as a primer to BCS theory.

To determine the energy spectrum $\epsilon(p)$ in a weakly interacting Bose gas at zero temperature we follow the approach by Bogoliubov. Starting from a Hamiltonian $\mathcal{H}_0$ of free bosons, perturbed by a weak interaction of the form

$$\mathcal{H}_{\text{int}} = \frac{1}{2V} \sum_{p+q=k+l} U_{p,q,k,l}\; a_p^\dagger a_q^\dagger a_k a_l$$

In the regime of a diluted gas at zero temperature the local interaction takes an s-wave form $U_{p,q,k,l} = U$ and the ground state is occupied by a large fraction of particles. The quartic interaction Hamiltonian can be simplified to

$$\mathcal{H}_{\text{int}} = \frac{U}{2V}\left[N^2 + \sum_{p\neq 0} N\left(2a_p^\dagger a_p + a_p^\dagger a_{-p}^\dagger + a_p a_{-p}\right)\right]$$

(a) The Bogoliubov transformation for the bosonic operators is given by

$$a_p = u_p \alpha_p + v_p \alpha_{-p}^\dagger$$
$$a_p^\dagger = u_p \alpha_p^\dagger + v_p \alpha_{-p}.$$

Verify that, in order for α_p to satisfy the bosonic commutation relations, the parameters $u_p, v_p \in \mathbb{R}$ should satisfy the constraint $u_p^2 - v_p^2 = 1$. A clever choice might then be

$$u_p = \frac{1}{\sqrt{1-K_p^2}} \qquad\qquad v_p = \frac{K_p}{\sqrt{1-K_p^2}}$$

(b) Express the full Hamiltonian $\mathcal{H}$ in these new bosonic excitations α_p. Find the appropriate choice of parameters u_p and v_p where $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{int}}$ is diagonal in α_p.

(c) Insert u_p and v_p back into the Hamiltonian and determine from this the excitation spectrum of the Bose gas.