**Exercise 1.** ***The Galilean and the Euclidean group***

In this exercise we are going to review the basic properties of some of the groups that appear in classical mechanics and that describe the invariance of physical laws with respect to transformation between coordinate systems. As seen in the lecture, for non-relativistic velocities $v \ll c$, different coordinate systems $S$ and $S'$ describing the same Euclidean spacetime $\mathbb{E} = \mathbb{R} \oplus \mathbb{R}^3$ are related by a Galilean transformation

$$\mathcal{G} : \mathbb{E} \to \mathbb{E}, \quad (t, \vec{r}) \mapsto (t', \vec{r}') = \mathcal{G}[(t, \vec{r})] \equiv (t + t_0, \boldsymbol{R}\vec{r} - \vec{r}_0 - \vec{V}t), \tag{1}$$

where $t_0$ is the time shown by the clock of $S'$ at time $t = 0$ in $S$, $\vec{r}_0$ is the translation of the origin of $S'$ with respect to the origin of $S$, and $\vec{V}$ is the apparent velocity of a point in $S'$ according to an observer in $S$, and $\boldsymbol{R}$ is a rotation of the axes of $S'$ with respect to the axes of $S$. Recall the definition of a group:

**Definition 1.** Given a set $G$ and a binary operation $\circ$: $G \times G \to G$, the algebraic structure $(G, \circ)$ is called a group if it satisfies the following requirements (group axioms):

1. Closure: $\forall a, b \in G \Rightarrow a \circ b \in G$
2. Associativity: $\forall a, b, c \in G \Rightarrow (a \circ b) \circ c = a \circ (b \circ c)$
3. Identity element: $\exists e \in G : \ \forall a \in G : e \circ a = a \circ e = a$
4. Inverse element: $\forall a \in G \ \exists a^{-1} \in G : a^{-1} \circ a = a \circ a^{-1} = e$

(a) i) Show that the set of all Galilean transformations $\mathcal{G}$, together with composition of transformations, forms a group SGal(3), called *(special) Galilean group.* Describe the general features of this group. How many parameters are necessary to completely describe it?

ii) Show that the three-dimensional rotations $\boldsymbol{R}$ form a proper subgroup SO(3) $<$ SGal(3).

*Hint. To show that the set $H$ is a subgroup of $G$, $H < G$, it is not necessary to check again all the group axioms for $H$. Instead it is sufficient to show that, for two elements $h_1$, $h_2 \in H$, the product $h_1 \circ h_2^{-1}$ always lies within the subspace. This is often called the "subgroup test".*

iii) Show that the composition of spatial translations $\mathbb{R}^3$ and proper rotations SO(3) forms a subgroup SEucl(3) $<$ SGal(3), called *(special) Euclidean group.* Can any element of SEucl(3) be written as a simple product of an element of $\mathbb{R}^3$ with and element of SO(3)?

*Hint. An element $\mathcal{S} \in$ SEucl(3) acts on $\mathbb{R}^3$ as follows:*

$$\mathcal{S} : \mathbb{R}^3 \to \mathbb{R}^3, \quad \vec{r} \mapsto \mathcal{S}(\vec{r}) \equiv \boldsymbol{R}\vec{r} - \vec{r}_0 \tag{2}$$

(b) The (Newtonian) principle of relativity states that the laws of classical mechanics are invariant under Galilean transformations. Does this principle extend to the laws of electrodynamics? Consider the Lorentz force

$$\vec{F}_L = q(\vec{E} + \vec{V} \times \vec{B}), \tag{3}$$

where $q$ is the charge, $\vec{E}$ is the electric field and $\vec{B}$ is the magnetic field.

i) How should the fields transform under a boost with velocity $\vec{V}_0$ to assure Galilean invariance of the Lorentz force for every reference frame?

ii) Now consider Gauss' law in vacuum:

$$\vec{\nabla} \cdot \vec{E} = 0. \tag{4}$$

Is this equation Galilean invariant?

**Exercise 2.** ***Soap film***

Consider a soap film suspended between two circular wires of radius $a$ parallel to the $yz$ plane and centred at $x = \pm\ell$. The film will adjust its shape such that the surface energy is minimised.

You should assume that the film is very thin, its shape is cylindrically symmetric and that the surface energy is given by $E = \sigma S$ where $\sigma$ is the surface tension and $S$ is the area of the film. Gravity should be ignored throughout this exercise. This means that the film eventually reaches a shape with minimal surface area.

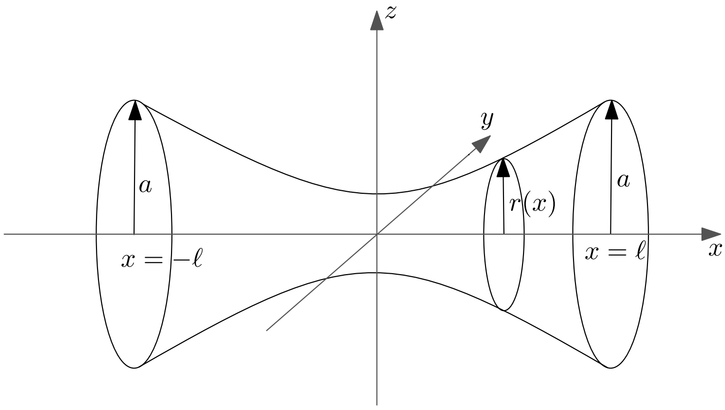


Figure 1: Soap film suspended between two circles

(a) Explain why the surface area can be written as

$$S[r] = \int_{-\ell}^{\ell} 2\pi r(x)\sqrt{1 + r'(x)^2}\,\mathrm{d}x \tag{5}$$

(b) Write down the Euler-Lagrange equations for minimising the functional $S[r]$ and show that they can be simplified to

$$\frac{r''r}{(1 + r'^2)^{3/2}} - \frac{1}{(1 + r'^2)^{1/2}} = 0. \tag{6}$$

Deduce that

$$\frac{r}{(1 + r'^2)^{1/2}} = c \tag{7}$$

for some constant $c$.

Note that in the steps above you first obtained a second-order differential equation and then transformed it into a much simpler first-order equation. There is actually a trick (known as the Beltrami identity) which allows you to write down immediately the first-order equation. In general, the functional to minimize/maximize is expressed as

$$T[y] = \int \mathrm{d}x \; F(y, y', x). \tag{8}$$

If $F(y, y', x) = F(y, y')$, i.e., the integrand in the functional does not depend explicitly on $x$, making use of Euler-Lagrange equations we will show that the quantity

$$H = F - y' \frac{\partial F}{\partial y'} \tag{9}$$

remains constant. This is proved by taking the total derivative

$$\frac{\mathrm{d}}{\mathrm{d}x} H = \frac{\partial F}{\partial y} y' + \frac{\partial F}{\partial y'} y'' - y'' \frac{\partial F}{\partial y'} - y' \frac{\mathrm{d}}{\mathrm{d}x} \left( \frac{\partial F}{\partial y'} \right) \tag{10}$$

(remember that $\frac{\partial F}{\partial x} = 0$, so this term is not included above). Now we use the Euler-Lagrange equations $\frac{\partial F}{\partial y} = \frac{d}{dx} \frac{\partial F}{\partial y'}$ and see that all terms above cancel. Hence

$$F - y' \frac{\partial F}{\partial y'} = c \,, \tag{11}$$

where $c$ is a constant. When we discuss Hamiltonian mechanics later in the class, we will see that the same result applies to an important physical quantity (the energy) when the variable $x$ is the time.

(c) Let us now go back to the soap film problem. Derive the equation (7) again, this time using the Beltrami identity. You should be able to obtain a solution to this equation of the form $r = c \cosh((x - x_0)/c)$. Using the boundary conditions show that $x_0 = 0$ and explain graphically why there is no solution if the ratio $a/\ell$ is smaller than a certain value. What happens physically in such case?

**Exercise 3.** ***Geodesics on a Sphere***

In this problem, we want to show that the path of shortest distance between two points on the surface of a sphere lies along the great circle that connects the two points. In general, curves of minimum path length between two points are called *geodesics.* They are generalizations of the definition of a straight line in a curved space.

(a) Using spherical coordinates, show that the path length from point $A$ to point $B$ is given by the following integral

$$l_{A \to B} = \int_A^B \mathrm{d}l = \int_A^B \sqrt{(\mathrm{d}r)^2 + r^2 (\mathrm{d}\theta)^2 + r^2 \sin^2 \theta (\mathrm{d}\phi)^2} \tag{12}$$

Since we want to find the path of least length on the surface of a sphere, we have the constraint $r = R$, where $R$ is the radius of the sphere.

(b) Using the constraint $r = R$ write the path length (12) in the following form

$$l_{A \to B} = R \int_{\phi_A}^{\phi_B} F[\theta(\phi), \theta'(\phi)] \mathrm{d}\phi \tag{13}$$

*Note that we chose to parametrize the path as $\theta(\phi)$ instead of the equivalent $\phi(\theta)$. Why?*

(c) Show that the curve which minimizes the path length is given by

$$\phi(\theta) = \arcsin(C_1 \cot\theta) + C_2, \tag{14}$$

where $C_1$ and $C_2$ are integration constant, which can be fixed by the requirement that the path passes through the points $A$ and $B$.

*Hint. Exploit the fact that the integrand* $F[\theta(\phi), \theta'(\phi)]$ *is independent of* $\phi$.

(d) Show that the solution (14) describes the great circle between points $A$ and $B$.

*Hint. Use the fact that a great circle is the intersection of a plane going through the origin of the coordinate system with the surface of the sphere.*