Exercise 1. *The Lie Algebra of SO(3)*

(a) Consider the rotation of a vector around the axis $\hat{n}$ by an angle $\delta\phi$:

$$\vec{r} \to \vec{r} + \delta\vec{r} + \mathcal{O}(\delta\phi^2) \tag{1}$$

where $\delta\vec{r}$ can be expressed as:

$$\delta\vec{r} = \delta\vec{\phi} \times \vec{r} \tag{2}$$

with

$$\delta\vec{\phi} = \hat{n}\delta\phi \tag{3}$$

Starting from Equation (2) with using Equation (3) find the generators of SO(3), the group of rotations.

(b) Compute the commutators of the generators and find the Lie algebra. Determine the structure constants and confirm that these also obey the Lie algebra.

(c) The generators can be written as the derivatives of the representation matrices with respect to the small transformation parameters:

$$(J_i)_{jk} = \frac{1}{i}\frac{\partial(R_i)_{jk}(\phi)}{\partial\phi}|_{\phi=0} \tag{4}$$

where R_i is the rotation matrix for a rotation about a generic i axis. Exponentiate the generators to find the representation matrices:

$$R_x(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\delta\phi & -\sin\delta\phi \\ 0 & \sin\delta\phi & \cos\delta\phi \end{pmatrix}, \quad R_y(\phi) = \begin{pmatrix} \cos\delta\phi & 0 & \sin\delta\phi \\ 0 & 1 & 0 \\ -\sin\delta\phi & 0 & \cos\delta\phi \end{pmatrix} \tag{5}$$

$$R_z(\phi) = \begin{pmatrix} \cos\delta\phi & -\sin\delta\phi & 0 \\ \sin\delta\phi & \cos\delta\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6}$$

Exercise 2. *Point moving on a paraboloid*

A point particle of mass m, subject to gravity, moves on a smooth paraboloid surface with equation $z = c^2(x^2 + y^2)$.

a) Write down the lagrangian for this system using cylindrical coordinates (r, ϕ, z) and expressing the constraint through Lagrange multipliers.

b) Use the rotational symmetry around the z-axis to work out the associated conserved quantity using Noether's theorem.

c) Write down the Euler-Lagrange equations and compare with point b). Deduce also that, for any fixed value of the conserved quantity, there is a value r_0 of the radial coordinate r that satisfies the remaining equation of motion with $\dot{r}(t) = 0$.

d) Expanding the equations around r_0 by means of $r(t) \equiv r_0 + \Delta r(t)$ with small Δr, find the motion of nearly circular orbits.

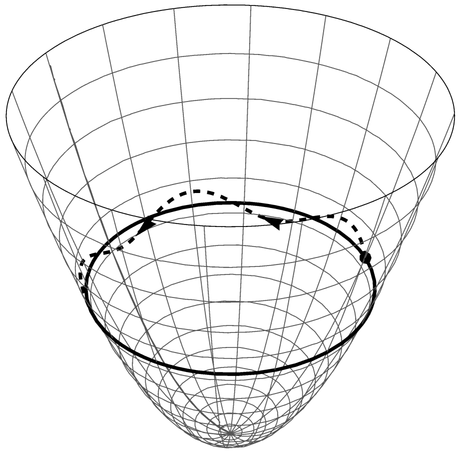

Figure 1: Exercise 2: point moving on a paraboloid.

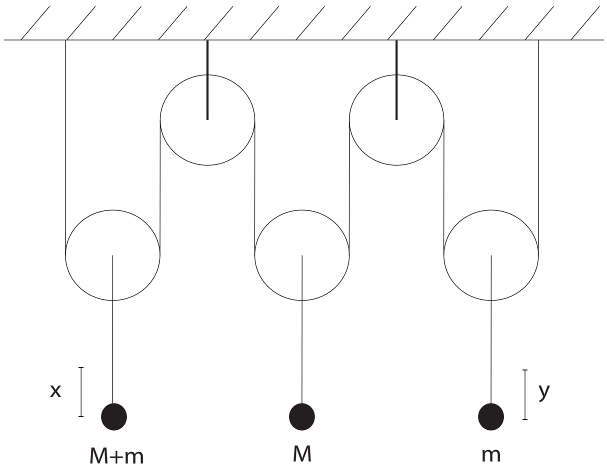

Figure 2: More sophisticated Atwood machine.

Exercise 3. ***Atwood Machine II***

Consider a more sophisticated Atwood machine as shown in Figure 2. Given the masses m, M and $m + M$ and the displacement coordinates x and y of the left and right masses as depicted, use Noether's theorem to derive the conserved momentum in this problem. Assuming that the system starts at rest, show that

$$\left(m^2 - 2M^2\right)\dot{x} = \left(M^2 + m^2\right)\dot{y}. \tag{7}$$