Exercise 1. ***Particle in a $1/r^n$ potential***

Consider a mass m moving on a circular orbit of radius r_0 under the influence of a central force whose potential is given by

$$V(r) = -\frac{km}{r^n} \tag{1}$$

where k is some constant. Show that if $n < 2$, the circular orbit is stable under small oscillations, *i.e.* the mass will oscillate about the circular orbit.

Exercise 2. ***The Central Drill***

Consider a satellite m orbiting around a planet of mass M, whose positions are $\vec{r}_1$ and $\vec{r}_2$ respectively. Let $\vec{R}$ be the position of the centre of mass of the planet-satellite system with $R \equiv |\vec{R}|$ and $\vec{r} = \vec{r}_1 - \vec{r}_2$ the relative coordinate with $r \equiv |\vec{r}|$. Also, let $k = GMm$, where G is Newton's gravitational constant.

(i) Making use of Newtonian mechanics show that the equations of motion are given by:

$$\mu\ddot{\vec{r}} = -k\frac{\vec{r}}{r^3} \quad \text{and} \quad \dot{\vec{R}} = \text{constant}$$

Find μ, the *reduced mass* of the system.

Hint: The centre of mass of a N body system is given by:

$$\vec{R} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i}$$

where m_i and $\vec{r}_i$ are the masses and positions of the bodies.

(ii) The Lagrangian of the system has been obtained in the lecture and reads

$$\mathcal{L} = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r), \tag{2}$$

where r, θ are polar coordinates in the relative reference frame. Rotate your system by a little amount:

$$\theta \to \theta + \epsilon$$

and use Noether's theorem or a careful observation of the Lagrangian to show that the angular momentum is conserved:

$$\mu r^2\dot{\theta} = \text{constant} = M.$$

(iii) About five centuries ago, Kepler figured out that the area swept by r in a given time is constant. Can you use conservation of angular momentum to conclude the same in 2015?

(iv) Verify that by taking the time derivative of the total energy you can recover the equation of motion in r.

Exercise 3. ***Symmetry of the orbit***

In the lecture we have seen that the two body problem can be reduced to a single body problem with reduced mass μ and a Lagrangian given by

$$\mathcal{L} = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r). \tag{3}$$

(a) Using conservation of angular momentum, show that the equation of motion for the radius can be expressed as:

$$\mu\ddot{r} - \frac{M^2}{\mu r^3} = -\frac{\partial V}{\partial r} \tag{4}$$

(b) Starting from conservation of angular momentum, convert equation (4) for $r(t)$ into the following equation for $r(\theta)$:

$$\frac{M^2}{\mu r^2}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\theta}\right) - \frac{M^2}{\mu r^3} = f(r), \tag{5}$$

where $f(r)$ is the conservative force due to the potential $V(r)$.

(c) Using the substitution $u = \frac{1}{r}$, show that the differential equation for the orbit satisfies $u(\theta) = u(-\theta)$. What does this imply for a practical construction of the orbit?

Exercise 4. ***Virial mass of clusters***

Among the applications of the virial theorem, there is an important one in astrophysics.
Consider an isotropic globular cluster of stars or galaxies, modelled as a uniform distribution of point-like masses m_i located at positions $\vec{r}_i$. Use the virial theorem to prove that, if one is able to measure the size and the distribution of velocities of the cluster, its mass M can be obtained as

$$M = \frac{5}{3}\frac{R\langle v^2\rangle}{G}, \tag{6}$$

where R is the radius of the cluster and $\langle v^2\rangle$ an appropriate mean square velocity. Typically, it is not possible to directly observe the velocities of the single stars or galaxies, which have to be inferred from Doppler shifts. What is actually measured is thus only the distribution of velocities along the line of sight. Re-express $\langle v^2\rangle$ in terms of the mean square Doppler velocity σ^2, and substitute it into the formula for M.