Exercise 1. ***Bar supported by springs***

A uniform horizontal thin bar of length L and mass M is supported at its ends by two vertical springs with force constants k_1 and k_2 respectively (see figure). At equilibrium, the springs have the same length and the bar lies horizontally. At $t = 0$ the bar is displaced from its equilibrium position and let free to move.

(a) Derive an expression for the moment of inertia of the bar about its center of mass.

(b) Choose a suitable set of generalised coordinates and find an expression for the Lagrangian.

(c) Linearise the system under the assumption of small oscillations and find the equations of motion. Assume that the center of mass of the bar stays on the same vertical line.

Hint. You can either start from the Lagrangian calculated in the previous question and Taylor expand it, assuming the motion around the equilibrium is small, or you can reformulate a new Lagrangian for the simplified problem and derive the equations of motions from it.

(d) Find the eigenmodes of the oscillations for the general case $k_1 \neq k_2$ and for the special case $k_1 = k_2 = k$.

Exercise 2. ***Satellite with antennae***

A cylindrical satellite of mass M and radius a is spinning about its axis with initial angular velocity $\bar{\omega}$. It carries within it two hollow antennae, each of mass m and length $2\,a$ lying one within the other along a diameter (the mass of the antennae it taken into account by the total mass of the satellite M). At the beginning these antennae are fully within the satellite, but they start being forced out **symmetrically** in opposite directions at constant speed, stretching a spring of strength constant K and natural length $2\,a$ which joins their ends.

(a) Set up the Lagrangian for the system assuming the spring has negligible mass (this involves the computation of the moment of inertia of the system as a function the position of the

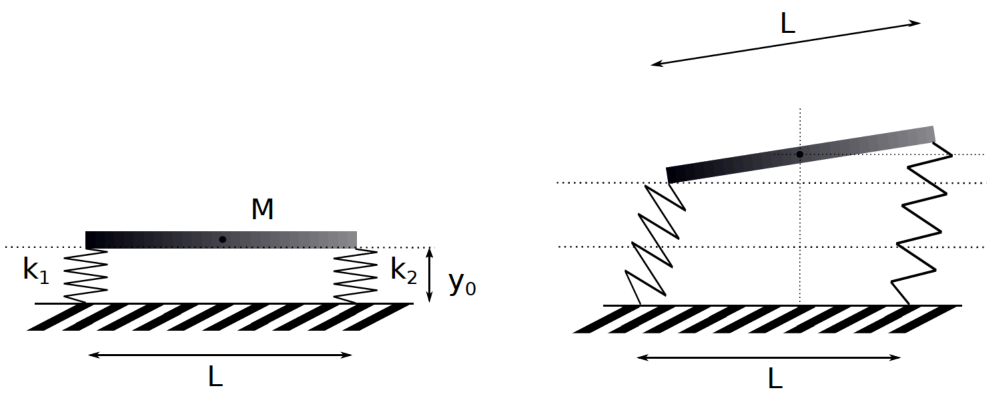

Figure 1: Sketch for the problem of a bar resting on two springs.

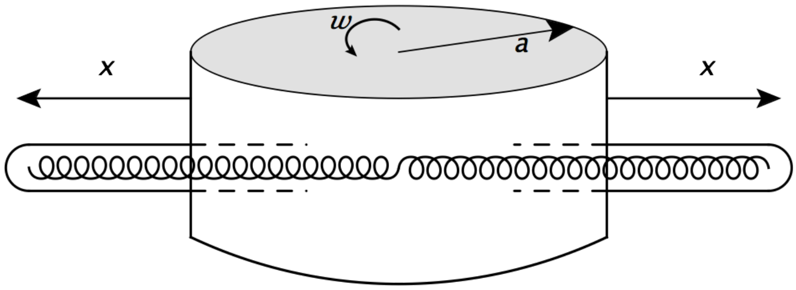

Figure 2: Sketch of the satellite with two antennae.

antennae).
Hint: describe the position of the antennae through the variable x, which is zero when the antennae are fully inside the satellite and $2a$ when they are fully outside.

(b) By using angular momentum conservation in the absence of external forces, find that the angular velocity of the system when the antennae are fully outside the satellite is

$$\omega(x = 2a) = \frac{M}{M + 16\,m}\bar{\omega} \tag{1}$$

(c) Show that it is possible to choose K so that no net work is done by the motor driving the antennae in fully extending them a distance $2\,a$ out of the satellite. Find that

$$K = \frac{1}{2}\frac{M\,m}{M + 16\,m}\,\bar{\omega}^2\,. \tag{2}$$

Exercise 3. ***The Isoperimetric Problem***

In this exercise we want to find among all closed curves with a given perimeter L, which one encloses the greatest area A?

(a) Parametrize the curves by $t \mapsto (x(t), y(t))$ with $t \in [0, 1]$ and show that

$$L = \int_0^1 \sqrt{\dot{x}^2 + \dot{y}^2}dt \qquad \text{and} \qquad A = \frac{1}{2}\int_0^1 (x\dot{y} - y\dot{x})dt, \tag{3}$$

where $\dot{x} = \frac{dx}{dt}$.

(b) Using the calculus of variations determine which curve maximizes the area A for a given perimeter L.

Hint. Note that we have a variational problem with an integral constraint. In order to see how to map this problem into an unconstraint one, check Exercise 3 (The Suspension Bridge) of Exercise Sheet 4.

Exercise 4. ***Rotating Asymmetric Top***

Show that the total torque on an object in homogeneous gravitational field vanishes if we choose the reference point in its centre of mass.

Consider an asymmetric object with principal moments of inertia $I_1 < I_2 < I_3$. It originally rotates completely about the first axis such that the angular velocities are $\omega_1 = \Omega$, $\omega_2 = \omega_3 = 0$. A small perturbation causes the angular velocities to change as

$$\omega_1 = \Omega + \eta_1(t),\ \omega_2 = \eta_2(t),\ \omega_3 = \eta_3(t).$$

Show from the Euler's equation by neglecting terms $\mathcal{O}(\eta^2)$ that the subsequent motion is described by

$$\ddot{\eta}_2 = A\eta_2$$

for some constant $A < 0$ that you should determine and find a similar equation for η_3. Conclude that the rotation about the first axis is stable. What happens if the object originally rotates about a different axis? Can you identify an axis for which the rotation is unstable?

You can now verify what you have found by taking your favorite book on classical mechanics and throwing it in the air.