Elasticity theory

Elastic energy – gradient expansion

Energy of a deformed Body can not depend on displacement itself because for uniform translation energy is not changed. Thus it can depend only on gradient of displacement $\frac{\partial u_i}{\partial x_k}$

For isotropic solid one can construct two quadratic invariants from $\frac{\partial u_i}{\partial x_k}$

$$\left(\frac{\partial u_i}{\partial x_i}\right)^2 = (\mathrm{div}\,\vec{u})^2 \quad \text{and} \quad \sum_{ik}\left(\frac{\partial u_i}{\partial x_k}\right)^2$$

However, not every gradient of displacement leads to energy change. The energy is also invariant with respect to uniform rotation

Infinitasimal rotation by an angle $\delta\Omega$

$$\vec{u} = \delta\vec{\Omega} \times \vec{r} \quad \text{with} \quad \delta\vec{\Omega} = \frac{\mathrm{rot}\,\vec{u}}{2}$$

Since energy should not depend on $\delta\vec{\Omega}$ thus from the gradient term we should exclude antisymmetric part and describe deformation by symmetric tensor $u_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)$

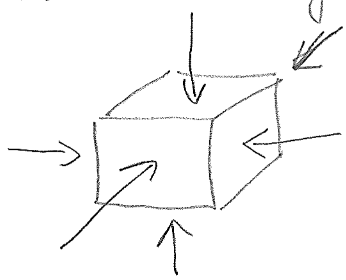
Then elastic energy is

$$F = \frac{\lambda}{2} u_{ii}^2 + \mu u_{ik}^2$$

λ, μ – Lamê coeffitients

We can diagonalise u_{ik} at any given paint

$$\begin{pmatrix} u_{11} & 0 & 0 \\ 0 & u_{22} & 0 \\ 0 & 0 & u_{33} \end{pmatrix}$$



Then $\vec{u}(x) = \vec{x}' - \vec{x}$, $dx'_i = dx_i + du_i$

$dx'_1 = (1 + u_{11})dx_1$, $dx'_2 = (1+u_{22})dx_2$, $dx'_3 = (1+u_{33})dx_3$

The volume $dV = dx_1 dx_2 dx_3 \Rightarrow$

$$dV' = dV(1+u_{11})(1+u_{22})(1+u_{33}) \approx dV\left(1 + \sum_i u_{ii}\right)$$

$dV' = dV(1 + u_{ii})$ (we sum over repeating indices)

The trace u_{ii} is invariant. Thus the relative volume change is

$$\frac{dV' - dV}{dV} = u_{ii} = \operatorname{div} \vec{u}$$

Stress tensor

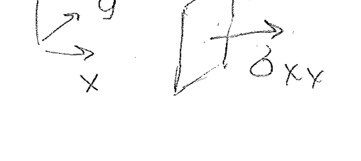
Before the deformation body is in equilibrum. =>
After deformation forces arise which tend to return the body to equilibrium. These forces are called <u>internal stresses</u>.

It is important that the internal stresses are due to molecular forces which have very short range of action. Thus in macroscopic elasticity theory these forces may be considered as having zero range of action (we do not consider here macroscopic electric fields in piezoelectrics).

Consider force acting on some volume  $\int \vec{F} dV$
The total force from the inner part is zero (Newton's third law). Thus the total force is the force from the neighbouring parts of the body which can be considered as applied to the surface =>

$$\int \vec{F}\, dV = \oint \sigma_{ik}\, dS_k \Rightarrow F_i = \frac{\partial \sigma_{ik}}{\partial x_k}$$

σ_{ik} – <u>stress tensor</u> (symmetric)



(σ_{ik} is i-th component of the force acting on the surface $\perp k$)

Hydrostatic compression $-p\,ds_i = -p\,\delta_{ik}\,ds_k \Rightarrow$

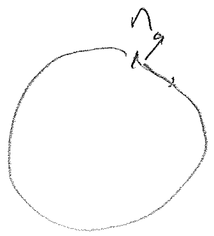
$\sigma_{ik} = -p\,\delta_{ik}$.

In equilibrium $\vec{F}_i = 0 \Rightarrow \dfrac{\partial \sigma_{ik}}{\partial x_k} = 0$

If we have external bulk forces (e.g. gravitation) $\Rightarrow$

$$\frac{\partial \sigma_{ik}}{\partial x_k} + \rho g_i = 0 \qquad (1)$$

If external forces are applied at the surface $\Rightarrow$



$P_i\,ds = \sigma_{ik}\,ds_k = \sigma_{ik} n_k\,ds$ ($\vec{n}$ - normal) $\Rightarrow$

boundary conditions $\sigma_{ik} n_k = P_i$

The work done under deformation

$$\int \delta R\,dV = \int F_i\,\delta u_i\,dV = \int \frac{\partial \sigma_{ik}}{\partial x_k}\,\delta u_i\,dV =$$

Integrating by parts $= \oint \sigma_{ik}\,\delta u_i\,ds_k - \int \sigma_{ik} \dfrac{\partial \delta u_i}{\partial x_k}\,dV$

Taking surface in the first integral to infinity where there are no deformation and using $\sigma_{ik} = \sigma_{ki}$ we obtain $\int \delta R\,dV = -\int \sigma_{ik}\,\delta u_{ik}\,dV \Rightarrow$

Change in the free energy $dF = -S\,dT - \delta R \Rightarrow$

$$\sigma_{ik} = \left(\frac{\partial F}{\partial u_{ik}}\right)_T$$

Hooke's law

In equilibrium $\sigma_{ik} = 0 = \dfrac{\partial F}{\partial u_{ik}} \Rightarrow$

Free energy F is quadratic in u_{ik}

From symmetric tensor one can make two quadratic scalars $\Rightarrow$

$$F = F_0 + \frac{\lambda}{2} u_{ii}^2 + \mu u_{ik}^2$$

λ, μ - Lamê coeffitients

If $u_{ii} = 0 \Rightarrow$ volume is unchanged $\Rightarrow$ shear

μ - shear modulus

$$u_{ik} = \overbrace{\left(u_{ik} - \frac{1}{3}\delta_{ik} u_{ee}\right)}^{\text{shear}} + \overbrace{\frac{1}{3}\delta_{ik} u_{ee}}^{\text{compression}}$$

$$F = \mu \left(u_{ik} - \frac{\delta_{ik} u_{ee}}{3}\right)^2 + \frac{K}{2} u_{ee}^2$$

K - compression modulus, $K = \lambda + \frac{2}{3}\mu$

If $u_{ee} = 0$ then only first term $F = \mu (\ldots)^2$

if $u_{ik} \propto \delta_{ik}$ then only the $\frac{K u_{ee}^2}{2} \Rightarrow$

$K > 0$, $\mu > 0$

$$dF = \left[K\, u_{ll}\,\delta_{ik} + 2\mu\left(u_{ik} - \frac{1}{3}\delta_{ik} u_{ll}\right)\right] du_{ik} \Rightarrow$$

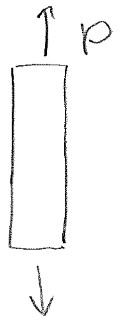
$$\sigma_{ik} = K\, u_{ll}\,\delta_{ik} + 2\mu\left(u_{ik} - \frac{1}{3}\delta_{ik} u_{ll}\right)$$

And inverting it we obtain

$$u_{ik} = \frac{1}{9K}\delta_{ik}\sigma_{ll} + \frac{1}{2\mu}\left(\sigma_{ik} - \frac{1}{3}\delta_{ik}\sigma_{ll}\right)$$

Hooke's law

Homogeneous deformations (strain tensor is constant)



extension

$$\sigma_{ik} n_k = p_i \Rightarrow$$

$\sigma_{zz} = p$ and other components $= 0$

$$u_{xx} = u_{yy} = -\frac{1}{3}\left(\frac{1}{2\mu} - \frac{1}{3K}\right) p$$

$$u_{zz} = \frac{1}{3}\left(\frac{1}{3K} + \frac{1}{\mu}\right) p = \frac{p}{E}$$

where $E = \dfrac{9K\mu}{3K+\mu}$ is the coefficient of extension or the Young's modulus

The ratio of the transverse compression to the longitudinal extension is called the Poisson's ratio

$$u_{xx} = -\sigma u_{zz}$$

$$\sigma = \frac{1}{2} \frac{3K - 2\mu}{3K + \mu}$$

Since $K, \mu > 0 \Rightarrow -1 < \sigma \leq \frac{1}{2}$

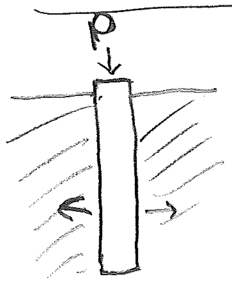
In reality, however $0 < \sigma \leq \frac{1}{2}$

$\sigma > 0 \Leftrightarrow \lambda > 0$.

Although not required from thermodynamics $\lambda > 0$ is realized in practice

$\sigma \to \frac{1}{2}$ corresponds to the situation, when $\mu \ll K$

Problem 1 Find deformation of a rod whose sides are fixed in such a way that they cannot move. Find the transverse force it exerts on the medium



In terms of E and σ free energy can be rewritten as

$$F = \frac{E}{2(1+\sigma)}\left(u_{ik}^2 + \frac{\sigma}{1-2\sigma} u_{ll}^2\right)$$

Then $\sigma_{ik} = \frac{E}{1+\sigma}\left(u_{ik} + \frac{\sigma}{1-2\sigma} u_{ll}\delta_{ik}\right)$ and inverting

$$u_{ik} = \frac{1}{E}\left[(1+\sigma)\sigma_{ik} - \sigma\delta_{ik}\sigma_{ll}\right]$$

Note, that since F is quadratic in in u_{ik} (Hooke's law) it can be rewritten as quadratic in σ_{ik}, then one can obtain

$$u_{ik} = \frac{\partial F}{\partial \sigma_{ik}}$$

One should emphasize, however, that this equation is consequence of the Hooke's law, whereas $\sigma_{ik} = \frac{\partial F}{\partial u_{ik}}$ is general relation of thermodynamics

Equations of equilibrium for isotropic bodies

Let us substitute expression for σ_{ik} in the general expression (1)

$$\frac{\partial \sigma_{ik}}{\partial x_k} + \rho g_i = 0 \Rightarrow$$

$$\frac{E}{2(1+\sigma)} \frac{\partial^2 u_i}{\partial x_k^2} + \frac{E}{2(1+\sigma)(1-2\sigma)} \frac{\partial^2 u_e}{\partial x_i \partial x_e} + \rho g_i = 0$$

Or in vector notations

$$\Delta \vec{u} + \frac{1}{1-2\sigma} \operatorname{grad} \operatorname{div} \vec{u} = -\rho \vec{g} \, \frac{2(1+\sigma)}{E} \quad (2)$$

Using $\operatorname{rot} \operatorname{rot} \vec{u} = \operatorname{grad} \operatorname{div} \vec{u} - \Delta \vec{u}$

we can exclude Δu and obtain

$$\operatorname{grad} \operatorname{div} \vec{u} - \frac{1-2\sigma}{2(1-\sigma)} \operatorname{rot} \operatorname{rot} \vec{u} = -\rho \vec{g} \, \frac{(1+\sigma)(1-2\sigma)}{E(1-\sigma)} \quad (3)$$

Some aplications

Let us determine deformation of a solid sphere of radius R in its own gravitational field (Earth)

Gravity force in a spherical body is $-g\vec{r}/R$

Substituting it into equation (3) we obtain

$$\frac{E(1-\sigma)}{(1+\sigma)(1-2\sigma)}\frac{d}{dr}\underbrace{\left(\frac{1}{r^2}\frac{d(r^2u)}{dr}\right)}_{\operatorname{div}\vec{u}} = \rho g \frac{r}{R}$$

Solution which is finite at $r=0$ and has $\sigma_{rr}=0$ at $r=R$ is

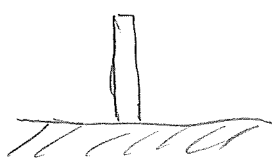
$$u = -\frac{g\rho R(1-2\sigma)(1+\sigma)}{10E(1-\sigma)}\, r\left(\frac{3-\sigma}{1+\sigma}-\frac{r^2}{R^2}\right)$$

Note, that the substance is compressed ($u_{rr}<0$) for $r < R\sqrt{(3-\sigma)/3(1+\sigma)}$ and stretched outside

At the center the pressure is $p=\frac{(3-\sigma)g\rho R}{10(1-\sigma)}$

Problem 2

Determine the deformation of the long rod standing vertically in a gravitation field



Elasticity of crystals

In general

$$F = \frac{1}{2}\lambda_{iklm} u_{ik} u_{lm} \quad , \quad \sigma_{ik} = \lambda_{iklm} u_{lm}$$

Since $u_{ik} = u_{ki}$ then

$$\lambda_{iklm} = \lambda_{kilm} = \lambda_{ikml} = \lambda_{lmik}$$

As a result of these symmetries in general there are 21 independent components. To see it note that ik can have six independent combinations. These are denoted by

$$1 = xx \,,\; 2 = yy \,,\; 3 = zz \,,\; 4 = yz \,,\; 5 = xz \,,\; 6 = xy$$

Then we can use $C_{\alpha\beta}$ instead of λ_{iklm} with $\alpha, \beta = 1, 2, \dots 6$.

For $\alpha = 1$ β can be any of 6 numbers. But since $C_{\alpha\beta} = C_{\beta\alpha}$ then $C_{12} = C_{21}$ and for $\alpha = 2$ β is $2, 3, \dots 6$ (5 numbers) for $\alpha = 3$ β is $3, 4, 5, 6$ etc. As a result $6+5+4+3+2+1 = 21$ independent components

Since orientation is given by the three angles then choosing appropriate frame reduces the number of components to 18

Example
For tetragonal system since

λ_{iklm} transforms as $x_i x_k x_l x_m$

Then because of symmetry

$x \to -x$, $y \to y$, $z \to z$ and

$x \to x$, $y \to -y$, $z \to z$ all components of λ_{iklm} with an odd number of like suffixes vanish.

Further more, a rotation through an angle $\frac{\pi}{2}$ $\to$

$x \to y$, $y \to -x$, $z \to z$ $\Rightarrow$

$\lambda_{xxxx} = \lambda_{yyyy}$, $\lambda_{xxzz} = \lambda_{yyzz}$, $\lambda_{xzxz} = \lambda_{yzyz}$

As a result there are 6 elastic constants

$$F = \frac{1}{2}\lambda_{xxxx}(u_{xx}^2 + u_{yy}^2) + \frac{1}{2}\lambda_{zzzz}u_{zz}^2 + \lambda_{xxzz}(u_{xx}u_{zz} + u_{yy}u_{zz})$$
$$+ \lambda_{xxyy}u_{xx}u_{yy} + 2\lambda_{xyxy}u_{xy}^2 + 2\lambda_{xzxz}(u_{xz}^2 + u_{yz}^2)$$

Problem 3

Show that for cubic system there are 3 independent elastic constants